SIMILARITY AND DIMENSIONAL METHODS IN MECHANICS

L. I. Sedov

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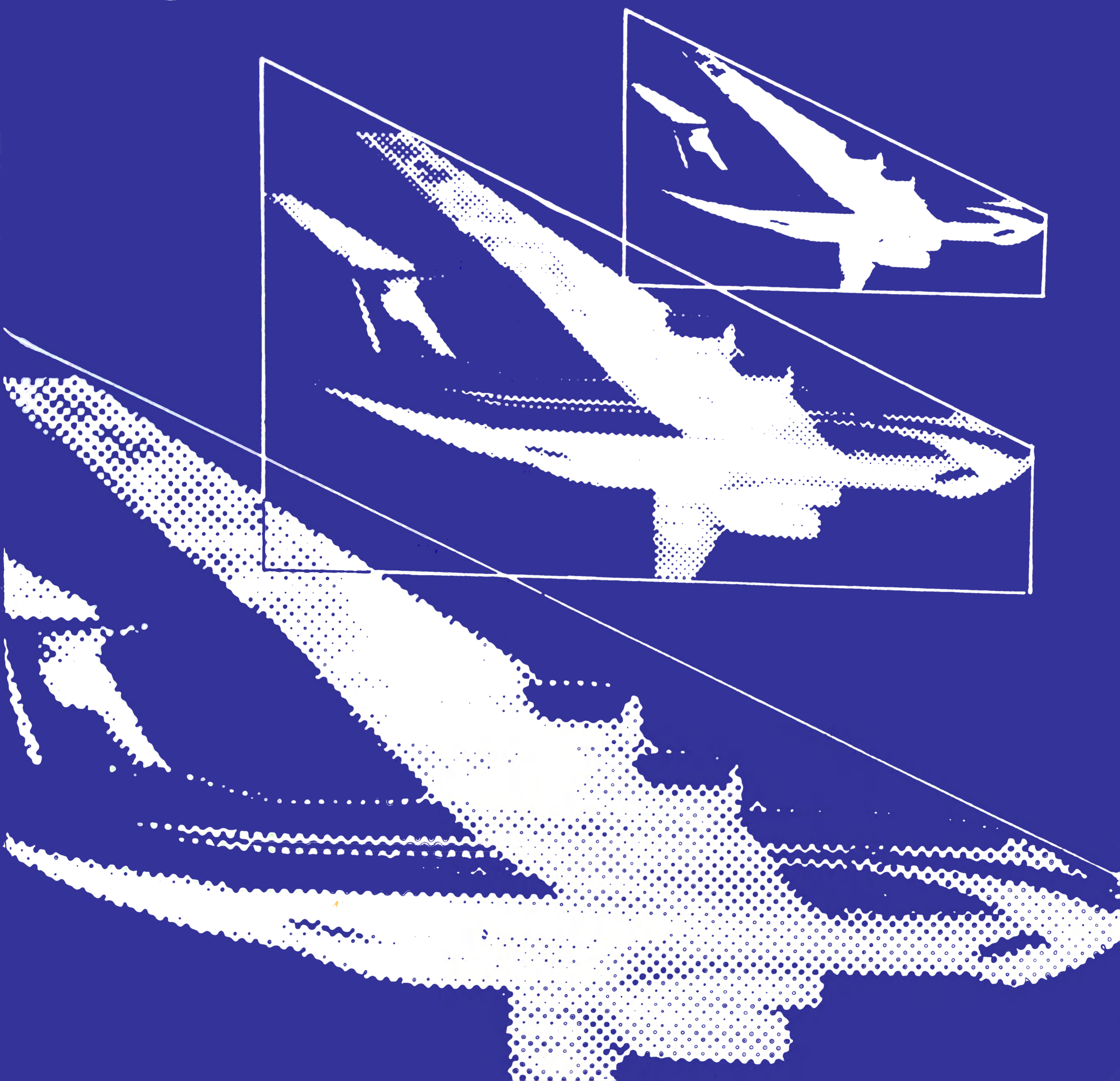

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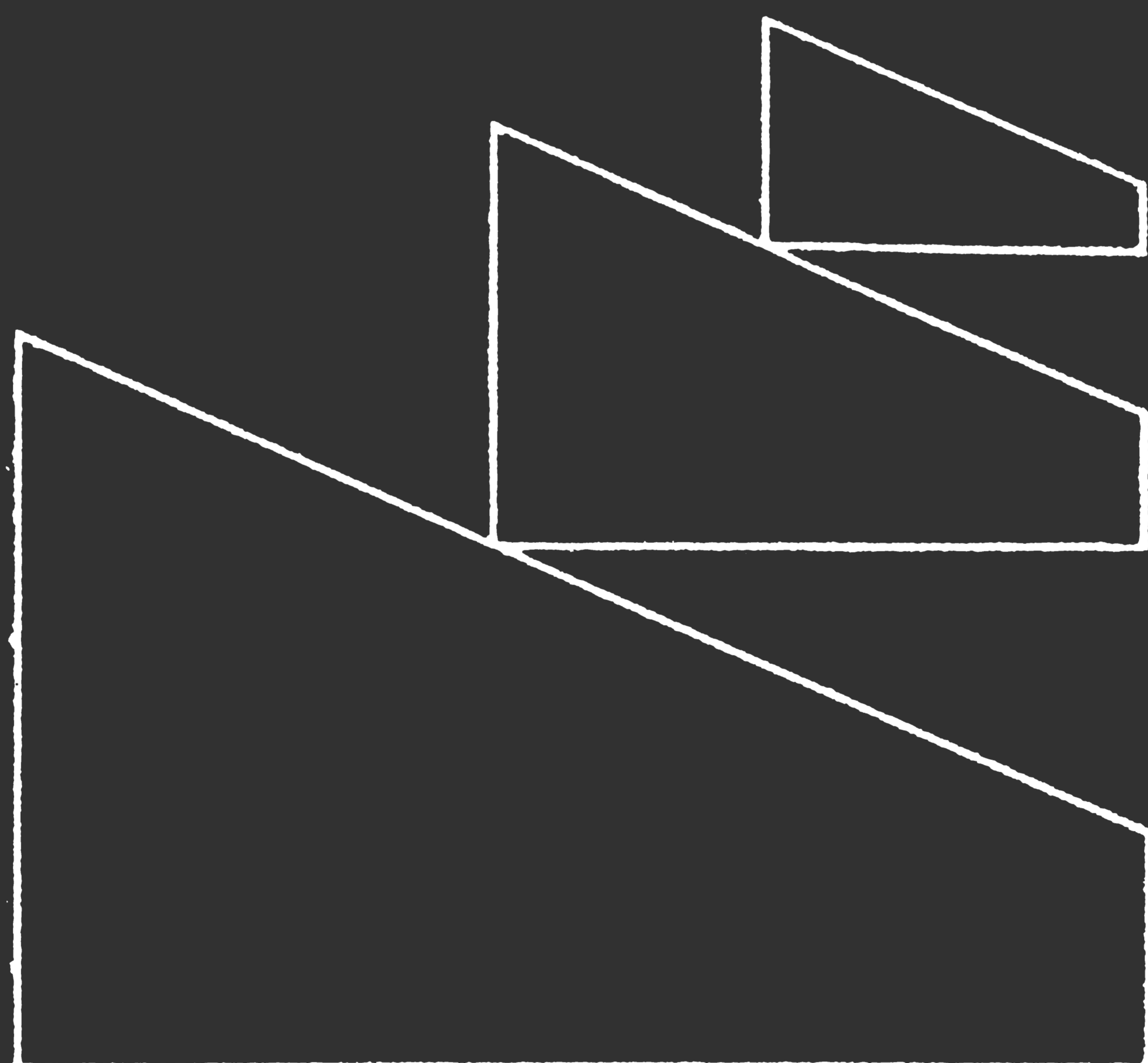

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Л. И. Седов

МЕТОДЫ ПОДОБИЯ И РАЗМЕРНОСТИ В МЕХАНИКЕ

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L. I. Sedov

SIMILARITY AND DIMENSIONAL METHODS IN MECHANICS

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CONTENTS

FOREWORD TO THE FIRST RUSSIAN EDITION

Physical similarity and dimensional properties play a very important part in experiments and calculations in physics and engineering. The construction of airplanes, ships, dams, and other complicated engineering structures is based on preliminary, broad investigations, including the testing of models. Dimensional analysis and similarity theory determine the conditions under which the model experiments are to be carried out and the key parameters representing fundamental effects and modes of operation. In addition, dimensional analysis and similarity theory when combined with the usual qualitative analysis of a physical phenomenon can be a fruitful means of investigation in a number of cases.

Dimensional analysis and the use of models are encountered in the earliest study of physics in schools and in the initial stages of formulating new problems in research work. Moreover, these theories are of an extremely simple and elementary character. In spite of this it is only in recent years that the reasonings of similarity theory have been widely and consciously used; in hydromechanics, for example, in the past thirty to forty years.

It is generally acknowledged that the explanation of these theories in textbooks and in actual teaching practice in colleges and universities is usually very inadequate; as a rule, these questions are only treated superficially or in passing. The fundamental concepts, such as those of dimensional and dimensionless quantities, the question of the number of basic units of measurement, etc., are not clearly explained. However, such confused and intuitive representations of the substance of the dimensions concept are often the origin of mystical or arcane physical imports attributed to dimensional formulas. In some cases, this vagueness has led to paradoxes which were a source of confusion. We shall examine in detail one example of such a misunderstanding in connection with Rayleigh's conclusions on heat emission from a body in fluid flow. Often, relations and mathematical techniques not related to the substance of the theory are used to

explain similarity theory. As in every general theory, it is desirable to construct the dimensional analysis and similarity theory by using methods and basic hypotheses which are adequate to the substance of the theory. Such a construction permits the limitations and possibilities of the theory to be clearly traced. This is necessary especially in dimensional analysis and similarity theory since they are regarded from widely different points of view: at one extreme they are considered to be all-powerful while at the other they are only expected to give trivial results. Both of these extreme opinions are incorrect.

However, it should be noted that similarity theory gives the most useful results when used in combination with general physical assumptions which do not in themselves yield interesting conclusions. Consequently, to show the range of application more completely, we consider a whole series of mechanical problems and examples in which we combine dimensional methods with other reasonings of a mechanical and a mathematical nature.

With this in mind, special attention is paid to the problems of turbulent fluid motion. Similarity methods are the basic techniques used in turbulence theory, since we still do not have a closed system of equations in this field which would permit the mechanical problem to be reduced to a mathematical one. New results are contained in the section on turbulent fluid motion which supplement and explain some aspects of turbulence theory. In addition to examples illustrating the use of methods of similarity and dimensional analysis, we discuss the formulation of a number of important mechanical problems some of which are new and hardly worked out.

We dwell in some detail on the analysis of the fundamental equation of mechanics derived from Newton's second law. This is of interest on its own account and also helps to illuminate the usual reasoning about basic mechanical properties. Our viewpoint on this matter is not new; however, it differs radically from the treatment given in certain widely used textbooks on theoretical mechanics.

The number of familiar applications of dimensional analysis and similarity theory in mechanics is very large; many of them are not touched upon here. The author hopes that the present book will give the reader an idea of standard methods and of their possibilities, which will be of assistance in the selection of new problems and in the formulation and treatment of new experiments.

A large part of the book does not require any special preparation by the reader. But in order to understand the material in the second half of the book, some general knowledge of hydromechanics is necessary.

Moscow, 1943 *L. I. Sedov*

FOREWORD TO THE THIRD RUSSIAN EDITION

In recent years, scientific investigations of physical phenomena have relied more and more on the invariant character of the governing mathematical and physical laws relative to the choice of units for measuring the physical variables and scales.

The practical and theoretical power of these methods has been recognized more and more by scientists contrary to the recently held opinion that the methods of similarity and dimensional analysis are of only secondary value.

A certain analogy exists between dimensional analysis and similarity theory and the geometric theory of invariants relative to coordinate transformation, a fundamental theory in modern mathematics and physics.

Since the first edition of this book appeared, many new applications of dimensional analysis and similarity theory have been made to widely different problems in physics and continuum mechanics, to certain mathematical problems related to the use of group theory in solving differential equations [1] and to statistical problems of sampling and inspection of goods and finished products [2].

Some corrections and additions to emphasize better the basic ideas of the theory of similarity and dimensional analysis are introduced in this edition. One example of this is the discussion of the proof of the Π-theorem. Furthermore, the definition of dynamic or physical similarity of phenomena has been given in more detail. This new definition is still not in general use in the similarity literature; however, from the practical viewpoint, it includes the essential peculiarities of physically similar processes; moreover, it is convenient for direct use and, apparently, satisfies all the needs of different applications.

Beyond this, §§ 8-12 in Chapter IV and an entirely new chapter have been added. The additions to Chapter IV are devoted to certain problems of explosions and the attenuation of shock waves, besides a discussion of the general theory of one-dimensional gas motion. Applications of the theory of one-dimensional unsteady gas motion

and the methods of dimensional analysis to certain astrophysical problems are considered in new Chapter V[1]).

The theory developed in the additions to Chapter IV and in Chapter V[1]) is completely new in its basic approach. The proposed formulation and solution of the gas dynamics problems illustrate the applications of dimensional analysis methods to astronomy and provide a stock of simple simulating ideal motions which can be used to investigate problems of cosmogony. Many of these results I obtained in collaboration with my young pupils in the course of the work of the hydromechanics seminar in Moscow University during the 1952-3 school year.

N.S. Mel'nikova and S.I. Sidorkina contributed to the preparation of § 14 of Chapter IV, V.A. Vasil'ev and M.L. Lidov to § 16, item 1° of Chapter IV, and I.M. Yavorskaya to § 6 of Chapter V[1]).

I express my deepest gratitude to them all.

Moscow, March 1954 *L. I. Sedov*

REFERENCES

[1]. Birkhoff, G., *Hydrodynamics. A Study in Logic, Fact, and Similitude*, Princeton University Press, 1950.

[2]. Drobot, S., Warmus, M., "Dimensional analysis in sampling inspection of merchandise", *Rozprawy Matematyczne*, V, Warszawa (1954).

FOREWORD TO THE SIXTH RUSSIAN EDITION

The sixth edition is supplemented with Chapter V, i.e. Introduction to the Theory of Gas Engines; some additions were made to Chapters I, IV, and VI. The texts of the earlier editions were checked out and a number of shortcomings was corrected.

References to recently published papers were enlarged. I shall specially single out a monograph *The Theory of Point Explosion* by V.P. Korobeinikov, N.S. Mel'nikova, and E.V. Ryazanov, which

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[1]) Chapter VI in the sixth and subsequent editions.

is a conceptual relative of the present book and gives a further elaboration of the theory of explosion.

I am very much indebted and deeply grateful to N.S. Mel'nikova who undertook the general editing and refinement of the text, as well as supervised additional computational work and preparation of new diagrams.

Moscow, February 1967 *L. I. Sedov*

FOREWORD TO THE EIGHTH RUSSIAN EDITION

As compared to the seven earlier editions, this one has a number of additions, comments, and improvements.

The topic that has been added to significantly is a comparison of the theory of isotropic turbulent flow (included in the first edition, i.e. in 1944) with the latest experimental results. Now that about forty years have elapsed, we have ample evidence that this theory, based on the dimensional analysis and similarity theory, is in good agreement with the experiments carried out within this period.

It can also be noted that the application-oriented aspects of the gas dynamics and dimensions theory, developed in Chapters IV and VI, gradually penetrate the realms of modern astrophysics and numerous other fields of science.

The arguments based on dimensions of various variable and constant quantities and on physical similarity (scaling) are widely used nowadays to formulate cognition problems as well as those in diverse fields of science and technology.

Moscow, May 1976 *L. I. Sedov*

FOREWORD TO THE NINTH RUSSIAN EDITION

The present, ninth, edition of the monograph has been purged of misprints that crept into the earlier ones, and supplemented with a considerable number of briefly annotated references to recent research results relevant to the body of the text on the theory of unsteady flow in continuous compressible media.

Apparently, not all significant contributions, especially those published outside of the USSR, are covered in the lists of References. I mostly cite the results connected with my line of work and contributing to the progress of the theories exposed in this monograph.

I hope that future will present us with history-oriented reviews which will fill this gap.

I wish to express my gratitude to V.V. Rozantseva who has prepared this edition for the publication and N.V. Morugina who did all the technical work with the manuscript.

Moscow October 1980 *L. I. Sedov*

CHAPTER I

GENERAL DIMENSIONS THEORY

§ 1. Introduction

Every phenomenon in mechanics is determined by a series of quantities, such as energy, velocity, and stress, which take on definite numerical values in specific cases.

Problems in dynamics or statics reduce to the determination of certain functions and characteristic parameters. The relevant laws of nature and geometrical relations are represented as functional equations, usually differential equations.

In purely theoretical investigations, we use these equations to establish the general qualitative properties of motion and to calculate unknown physical quantities using mathematical techniques. However, it is not always possible to solve a mechanical problem solely by the processes of analysis and calculation; sometimes the mathematical difficulties are too great. Very often the problem cannot be formulated mathematically because the mechanical phenomenon to be investigated is too complex to be described by a satisfactory model and the equations of motion are unknown. This situation arises in many important problems in aeromechanics, hydromechanics, and the theory of structures; in these cases, we have to rely mainly on experimental methods of investigation to establish the essential physical features of the problem. In general, we begin every investigation of a natural phenomenon by finding out which physical properties are important and by looking for mathematical relations between them which govern the phenomenon.

Many phenomena cannot be investigated directly, and to determine the laws governing them we must perform experiments on similar phenomena which are easier to handle. To set up the most suitable experiments we must make a general qualitative analysis and bring out the essentials of the phenomenon in question. Moreover, theoretical analysis is needed when formulating experiments to determine the values of particular parameters of the phenomenon. In general, and particularly in designing experiments, it is very important to select the dimensionless parameters correctly; there should be as few parameters as possible and they must reflect the fundamental effects in the most convenient way.

This preliminary analysis of a phenomenon and the choice of a system of principal dimensionless parameters are made possible by dimensional analysis and similarity theory: it can be used to analyse very complex phenomena and is of considerable help in processing experimental data. In fact it is out of the question to formulate and carry out experiments nowadays without making use of similarity and dimensions concepts. Sometimes dimensional analysis is the only theoretical means available at the beginning of an investigation of some phenomenon. However, the potentialities of the method should not be overestimated. In many cases, only very limited or trivial results are obtained from dimensional analysis. On the other hand, the widely held opinion that dimensional analysis rarely yields results of any importance is completely unjustified: quite significant results can be obtained by combining similarity theory with the data obtained from experiment or from the mathematical equations of motion. In general, dimensional analysis and similarity theory are very useful both in theory and practice. All results derived from this theory are obtained in a simple and elementary manner. Nevertheless, in spite of their simple and elementary character, the methods of dimensional analysis and similarity theory require considerable experience and ingenuity on the part of an investigator when probing into the properties of some new phenomenon [1].

In the study of phenomena which depend on a large number of parameters, dimensional analysis is especially valuable in determining which parameters are irrelevant and which are significant. We shall illustrate this point later by examples. The methods of dimensional analysis and similarity theory play an especially large part in simulating various phenomena.

§ 2. Dimensional and Dimensionless Quantities

Quantities are called dimensional or concrete if their numerical values depend on the scale used, that is, on the system of the units of measurement. Quantities are called dimensionless or abstract if their values are independent of the system of units. Typical dimensional quantities are length, time, force, energy, and moment. Angles, the ratio of two lengths, the ratio of the square of a length to an area, the ratio of energy to moment, etc. are examples of dimensionless quantities.

However, the subdivision of quantities into dimensional and dimensionless is to a certain extent a matter of convention. For example, we have just called an angle dimensionless. It is known that angles can be measured in various units, such as radians, degrees, or fractions of a right angle. Therefore, the number defining an angle depends on the choice of units; consequently, an angle can be considered a dimensional quantity. Suppose we define an angle as

the ratio of the subtended arc of a circle to its radius; the radian—the angular unit—will then be defined uniquely. Now, if an angle is measured only in radians in all systems of units, then it can be considered a dimensionless quantity. Exactly the same argument applies if a single fixed unit of measurement is introduced for length in all systems of units. In these circumstances length can be considered dimensionless. But it is convenient to fix the unit for angle and inconvenient for length: this is explained by the fact that corresponding angles of geometrically similar figures are identical while corresponding lengths are not and, consequently, it is convenient to use different basic lengths in different problems.

Acceleration is usually considered a dimensional quantity with the dimensions of length divided by time squared. In many problems, the acceleration due to gravity g, equal to the acceleration of a body falling in a vacuum, can be considered constant (9.81 m/s^2). This constant acceleration g can be selected as a fixed unit of measurement for acceleration in all systems of units. Then any acceleration will be measured by the ratio of its magnitude to the magnitude of the acceleration due to gravity. This ratio is called the load factor, a numerical value of which will not vary when converting one unit to another. Therefore, the load factor is a dimensionless quantity. But the load factor can be considered a dimensional quantity at the same time, namely, as acceleration when the acceleration due to gravity is taken as the unit. In this latter case, we assume that the load factor—the acceleration—can be taken as a unit which is not equal to the acceleration due to gravity.

On the other hand, abstract (dimensionless) quantities can be expressed in various numerical forms. In fact, the ratio of two lengths can be expressed as an arithmetic quotient, as a percentage, or by other means.

The concepts of dimensional and dimensionless quantities are therefore relative. A certain excess of units is employed. When these units are identical in all systems, the corresponding quantities are called dimensionless. Dimensional quantities are defined as those for which the units can vary in experimental or in theoretical investigations. Here it is irrelevant whether or not the investigations are actually carried out. It follows from this definition that certain quantities can be considered dimensional in some cases and dimensionless in others. We gave examples of these above and later we shall encounter a number of others.

§ 3. Fundamental and Derived Units of Measurement

Different physical quantities are interrelated via a number of relationships. Therefore, if certain physical quantities are taken as basic with assigned units, then the units of measurement of all the

remaining quantities can be expressed in a definite manner in terms of those of the fundamental quantities. The units taken for the fundamental quantities will be called fundamental or primary, and all the rest will be derived or secondary.

In practice, it is sufficient to establish the units for three quantities; precisely which three depends on the particular conditions of a problem. Thus, in physical investigations it is convenient to take the units of length, time, and mass as the fundamental units, and in engineering investigations to take the units of length, time, and force. But the units of velocity, viscosity, and density, etc. could also be taken as the fundamental units.

At the present time, the physical and absolute mechanical systems of units have become most widespread. The centimetre, gram, and second have been adopted as the fundamental units in the physical system (hence the abbreviation—cgs system of units). The metre, kilogram-force, and second have been adopted as the fundamental units in the absolute mechanical system (hence the abbreviation—mks system of units).

The units of length, the metre (equal to 100 cm), of mass, the kilogram (equal to 1000 g), and of time, the second, have been established experimentally by definite agreement. Until 1960 the length of a bar of platinum-iridium alloy, stored in the French Bureau of Weights and Measures, was taken as the metre; the mass of another bar of platinum-iridium alloy, stored in the same Bureau, was taken as the kilogram. The second was assumed to be $1/(24 \times 3600)$ part of a mean solar day [2].

A system of units which is becoming more and more widespread is the unified International System of Units, SI (from the French, Système International d'Unités). It was enacted as a mandatory system in the USSR in 1963, and in the COMECON as a whole in 1979.

The fundamental mechanical units in the SI are: the metre, kilogram of mass, and second; the unit of current is the ampere, that of the thermodynamic temperature is the kelvin, the unit of the luminous intensity is the candela, and that of the amount of substance is the mole [3].

Once the fundamental units have been established, the units for the other mechanical quantities, such as force, energy, velocity, and acceleration, are obtained automatically from their definitions.

The expression of the derived units in terms of the fundamental units is called their dimensions. The dimensions are written as a formula in which the symbol for the dimensions of length, mass, and time is denoted by L, M, and T, respectively (in the absolute mechanical system, the unit of force is denoted by K). When discussing dimensions, we must use a fixed system of units. For example, the dimensions of area are L^2; the dimensions of velocity are L/T or LT^{-1}, the

dimensions of force in the physical system are ML/T^2 and in the absolute mechanical system, K.

We shall use the symbol $[a]$ to denote the dimensions of any quantity a. (Maxwell was the first to use this notation.) For example, we shall write for the dimensions of force F in the physical system: $[F] = ML/T^2$ or $ML/T^2 = K$.

Dimensions formulas are very convenient for converting the numerical values of a dimensional quantity when the units are converted from one system to another. For example, in measuring the acceleration due to gravity in centimetres and seconds, we have: $g = 981\ \text{cm/s}^2$. If we need to convert from these units to kilometres and hours, then the following relations should be used to convert the numerical value of the acceleration due to gravity:

$$1\ \text{cm} = \frac{1}{10^5}\ \text{km}, \quad 1\ \text{s} = \frac{1}{3600}\ \text{h}$$

hence

$$g = 981\ \frac{\text{cm}}{\text{s}^2} = 981\ \frac{\frac{1}{10^5}\ \text{km}}{\left(\frac{1}{3600}\right)^2 \text{h}^2} = 98.1 \times 36^2\ \frac{\text{km}}{\text{h}^2}$$

In general, if the units of length, mass, and time in the new system of units are reduced by factors of α, β, and γ, respectively, in terms of the corresponding units in the old system, then the numerical value of the physical quantity a, with the dimensions $[a] = L^l M^m T^n$, is increased by a factor of $\alpha^l \beta^m \gamma^n$ in the new system.

The number of fundamental units need not necessarily equal three: a greater number of units can be taken. For example, the units for four quantities, that is, length, time, mass, and force, can be independently established by experiment. The Newton equation becomes in this case

$$F = cma$$

where F is the force, m is the mass, a is the acceleration, and c is a constant with dimensions $[c] = KT^2/(ML)$.

In the general case, four arguments will enter into the dimensions formulas of the mechanical quantities when the fundamental units are chosen in this way. The constant c in the above equation is a physical constant similar to the acceleration due to gravity g or to the gravitational constant G in the law of gravitation

$$F = G\,\frac{m_1 m_2}{r^2}$$

where m_1 and m_2 are the masses of two particles, and r is the distance between them. The numerical value of the constant c will depend on the choice of the fundamental units.

If the constant c is considered an abstract number (so that c will have the same numerical value in all systems of units) not necessarily equal to unity, then the dimensions of force are defined in terms of mass, length, and time, and the unit of force will be defined uniquely in terms of the units of mass, length, and time.

In general, we can tentatively select independent units for n quantities ($n > 3$) provided that we introduce $n - 3$ additional dimensional physical constants at the same time. In this case, the formulas of the derived quantities will generally contain n arguments.

When studying mechanical phenomena, it is sufficient to introduce only three independent fundamental units, that is, for length, mass (or force), and time. These units can also be used in studying thermal and even electrical phenomena. It is known from physics that the dimensions of thermal and of electrical quantities can be expressed in terms of L, M, and T. For example, the quantity of heat and temperature have the dimensions of mechanical energy. However, in many questions of thermodynamics and of gas dynamics, it is customary, in practice, to select the units for the quantity of heat and for temperature independently of the units for mechanical energy. The unit used to measure temperature is the degree Celsius or kelvin, and to measure the quantity of heat we use the calorie. These units have been established experimentally, independently of the units for mechanical quantities.

When studying phenomena in which a conversion of mechanical energy into heat occurs, it is necessary to introduce two additional physical dimensional constants; one of these is the mechanical equivalent of heat, or Joule's equivalent, $J = 427$ kgf-m/kcal and the other is either the specific heat c cal/m$^3\cdot$K, the gas constant $R = 8.31 \times 10^7$ erg$\cdot$mol$^{-1}\cdot$K^{-1}, or the Boltzmann constant $k = 1.38 \times 10^{-16}$ erg$\cdot$K^{-1}. If we wish to measure the quantity of heat and temperature in mechanical units, then the mechanical equivalent of heat and the Boltzmann constant will enter into the formulas as absolute dimensionless constants and will be similar to conversion factors in changing, for example, metres into feet, ergs into kilogram-metres, etc.

It is not difficult to see that fewer than three fundamental units can be taken. In fact, we can compare all forces with gravity, although this is inconvenient and unnatural when gravitation plays no part. Force in the physical system of units is generally defined by the equality

$$F = ma$$

and gravity by

$$F' = G\frac{m_1 m_2}{r^2}$$

where G is the gravitational constant with the dimensions $[G] = M^{-1}L^3T^{-2}$. When measuring heat in mechanical units, the dimen-

sional constant of the mechanical equivalent of heat can be replaced by a dimensionless constant. In the same way the gravitational constant can be considered an absolute dimensionless quantity. The dimensions of mass can then be expressed in terms of L and T by the relation: $[m] = \mathrm{M} = \mathrm{L}^3\mathrm{T}^{-2}$. Therefore, the charge in the unit mass in this case is determined completely by the charge in the units of length and time. Hence, if we regard the gravitational constant as an absolute dimensionless constant, we shall have the total of two independent units.

The number of independent units can be reduced to one if we regard some dimensional physical quantity, such as the coefficient of kinematic viscosity of water, ν, or the velocity of light in a vacuum, c, as an absolute dimensionless constant.

Finally, we can consider all physical quantities to be dimensionless if we regard appropriate physical quantities as absolute dimensionless constants. In this case, the possibility of using different systems of units is ruled out. The single system of units obtained is based on the physical quantities selected (for example, on the gravitational constant, the velocity of light, and the coefficient of viscosity of water), their values being taken as absolute universal constants.

There is a tendency to introduce such a system in scientific investigations since it permits the establishment of units of a permanent character. In contrast, the standards for the metre and the kilogram are essentially accidental quantities unrelated to the fundamental phenomena of nature [4].

The introduction of a single system of units excluding all other systems is equivalent to abandoning the dimensions concept completely. The numerical values of all characteristic quantities in a single universal system of units determine their physical magnitude uniquely.

A single universal system of units of this type, that is, the use of identical measures, methods of calculating time, etc., would have certain definite advantages in practice since it would be one of the links standardizing measurement methods.

However, in many phenomena, such special constants as the gravitational constant, the velocity of light in a vacuum or the coefficient of kinematic viscosity of water are completely irrelevant. Consequently, a single universal system of units related to the laws of gravitation, light propagation, and viscous friction in water or to any other physical processes would often be artificial and impractical. Since phenomena in different branches of physics are of such a widely varied character, it is desirable to be able to carry the system of units adjusted to the conditions peculiar to each specific domain.

It is convenient to choose force, length, and time as fundamental units in mechanics, using different units of force and length in celestial mechanics; it is more suitable to take the current intensity,

resistance, length, and time as fundamental units in electrical engineering (ampere, ohm, centimetre, and second), etc.

Moreover, the numerical values of the characteristic quantities arising in the study of a particular phenomenon are often expressed advantageously as ratios to the most significant parameters in that phenomenon. These fundamental characteristic quantities can differ from one case to another.

§ 4. Dimensions Formulas

The relation between the units of the derived quantities and those of the fundamental quantities can be represented as a formula. This formula is called the dimensions formula and can be considered a compact definition and description of the physical nature of the derived quantity.

Dimensions can only be understood in application to a definite system of units. The dimensions formula for the same quantity in different systems of units may contain a different number of arguments and have different forms. The dimensions formulas of all the physical quantities in the cgs system of units have the form of a power monomial $L^l M^m T^t$. We deduce this result from the physical property that the ratio of two numerical values of any derived quantity must be independent of the choice of the scale for the fundamental units. For example, suppose that we measure area, first, in square metres and, second, in square centimetres, then the ratio of two different areas measured in square metres would be the same as the ratio of the same areas measured in square centimetres. This condition imposed on the fundamental quantities is implied by the definition of the units.

We consider any derived dimensional quantity y; for simplicity, let us first assume that y is geometric and, consequently, depends only on length. Then

$$y = f(x_1, x_2, \ldots, x_n)$$

where $x_1, x_2, \ldots, x_n$ are certain distances. Let us denote the value of y corresponding to the values of the arguments $x'_1, x'_2, \ldots, x'_n$ by y'. The numerical value of y, as well as of y', depends on the unit for the distances $x_1, x_2, \ldots, x_n$. Let us diminish this unit or length scale by a factor of α. According to the condition formulated above, we should then have:

$$\frac{y'}{y} = \frac{f(x'_1, x'_2, \ldots, x'_n)}{f(x_1, x_2, \ldots, x_n)} = \frac{f(x'_1\alpha, x'_2\alpha, \ldots, x'_n\alpha)}{f(x_1\alpha, x_2\alpha, \ldots, x_n\alpha)} \tag{4.1}$$

i.e. the y'/y ratio must be identical for any value α of the length scale. We obtain from equation (4.1):

$$\frac{f(x_1\alpha, x_2\alpha, \ldots, x_n\alpha)}{f(x_1, x_2, \ldots, x_n)} = \frac{f(x'_1\alpha, x'_2\alpha, \ldots, x'_n\alpha)}{f(x'_1, x'_2, \ldots, x'_n)}$$

or

$$\frac{y(\alpha)}{y(1)} = \frac{y'(\alpha)}{y'(1)} = \varphi(\alpha) \tag{4.2}$$

Therefore, the ratio of the numerical values of the derived geometric quantities measured in different length scales depends only on the ratio of the length scales.

It is easy to find the form of the function $\varphi(\alpha)$ from relation (4.2). In fact, we have

$$\frac{y(\alpha_1)}{y(1)} = \varphi(\alpha_1), \qquad \frac{y(\alpha_2)}{y(1)} = \varphi(\alpha_2)$$

Hence, we obtain:

$$\frac{\varphi(\alpha_1)}{\varphi(\alpha_2)} = \varphi\left(\frac{\alpha_1}{\alpha_2}\right) \tag{4.3}$$

since we have for $x_1' = x_1\alpha_2$, $x_2' = x_2\alpha_2$, . . ., $x_n' = x_n\alpha_2$

$$\frac{y(\alpha_1)}{y(\alpha_2)} = \frac{y_1'\left(\frac{\alpha_1}{\alpha_2}\right)}{y_1'(1)} = \varphi\left(\frac{\alpha_1}{\alpha_2}\right)$$

Differentiating equation (4.3) with respect to α_1 and putting $\alpha_1 = \alpha_2 = \alpha$, we obtain

$$\frac{1}{\varphi(\alpha)}\frac{d\varphi}{d\alpha} = \frac{1}{\alpha}\left(\frac{d\varphi(\alpha)}{d\alpha}\right)_{\alpha=1} = \frac{m}{\alpha}$$

Integrating, we find

$$\varphi = C\alpha^m$$

Since $\varphi = 1$ when $\alpha = 1$, then $C = 1$; therefore,

$$\varphi = \alpha^m \tag{4.4n}$$

This result holds for any dimensional quantity depending oy several fundamental quantities if we vary just one scale. It is eass to see that if the α, β, and γ scales of the three fundamental quantitie are altered, then the function φ will be

$$\varphi = \alpha^m\beta^n\gamma^t$$

This proves that the dimensions formulas of physical quantities must be power monomials.

§ 5. On Newton's Second Law

When investigating mechanical or physical phenomena, we introduce, first, a system of quantities characterizing the various aspects of the processes being studied (let us call them simply characteristic quantities) and, second, a system of units that is used to determine the numerical values of these quantities. A number of

relations exist between the characteristic quantities of phenomena. Some of these relations apply only to a specific system and to a particular part of the process, other relations may be valid for certain classes of systems and motions. The relations of the latter type have special value and to look for them is an important object of physical investigations.

The methods of similarity and dimensional analysis provide one way of determining relations between characteristic quantities. In what follows we intend to show ways and means of applying these methods. But before explaining them directly, we shall illustrate certain important aspects by means of examples. In this connection, we consider the fundamental relation of mechanics known as Newton's second law.

Certain relations between characteristic quantities are simple consequences of their definition. For example, the magnitude of the velocity v equals the ratio of the path travelled to the corresponding time interval, the magnitude of the kinetic energy E of a particle equals $mv^2/2$, where m is the mass of the particle, and so on.

In addition to these trivial relations, theoretical and experimental investigations establish functional relations between the numerical values of characteristic quantities. These reflect the peculiar properties of the phenomenon, or class of phenomena, under consideration. The Kepler laws of planetary motion and the law of gravitation are examples of such relations. Let us illustrate the relation between these laws briefly.

As the result of observing the motion of planets extensively over many years, Kepler formulated the following general laws in 1609 and 1619:

(1) The planets describe ellipses around the sun, and the sun is always at one of the foci.

(2) The radius-vector connecting the sun with a planet sweeps over equal areas in equal time intervals.

(3) The square of the period of rotation of a planet around the sun is proportional to the cube of the corresponding mean distance of the planet from the sun.

If the magnitude of the force of interaction between the sun and a planet is defined as the mass multiplied by the acceleration, then the law of gravitation can be derived mathematically from the Kepler laws, namely,

$$F = G \frac{m_1 m_2}{r^2} \tag{5.1}$$

where F is the force of attraction, r is the distance between two particles of masses m_1 and m_2. This law was established by Newton in 1682 and was subsequently checked and verified by a comparison of the numerous results obtained from it with observations in nature and in special experiments.

Another example is Hooke's law which relates the spring tension F with its extension x.

This law is derived from static and dynamic observations of a load suspended on a spring, using the definition of the magnitude of force as the product of mass by acceleration and, sometimes, the rule of addition of forces.

In mathematical terms, this law is written as

$$F = kx \tag{5.2}$$

where k is the spring constant.

Using this law, the law of motion (i.e. the expression of all the mechanical quantities as functions of time), the period of oscillations, etc. can be determined theoretically in a number of different special cases (suspending the load on several springs, varying the mass or the spring constant, varying the initial conditions, etc.).

The solution of these and of other similar problems of mechanics is based on the investigation of the equation of motion of a particle

$$\mathbf{F} = m\mathbf{a} \tag{5.3}$$

where $\mathbf{a}$ is the acceleration vector, m is the mass of the particle, and $\mathbf{F}$ is the force vector. Very often the force $\mathbf{F}$ is the vector sum of several forces

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots \tag{5.4}$$

representing the total of a series of effects. The possibility of replacing several forces acting simultaneously by one force, defined by formula (5.4), is an experimental fact.

Now, let us consider the quantities in (5.3) in more detail. The acceleration $\mathbf{a}$ is a kinematic quantity which can always be determined experimentally, independently of (5.3). The mass m defines the inertial property of a body. The idea of the mass of a particle can be introduced by means of Newton's third law (every action has an equal and opposite reaction). In fact, the value of a constant magnitude, its mass, can be attributed to each particle, so that the following relations will hold for the motion of any two isolated interacting particles M_1 and M_2 or M_1 and M_3:

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0, \quad m_1\mathbf{a}_1' + m_3\mathbf{a}_3' = 0 \tag{5.5}$$

Therefore, the ratio of the masses can always be determined experimentally, independently of (5.3), by measuring the ratio of the accelerations when the interacting bodies move.

The constancy of mass, defined by (5.5) for every kind of motion, is an experimental fact expressing a law of nature which, in general, can be made more precise.

If the motion is known, then (5.3) is a simple equation to determine the value of the total force. In practice, (5.3) is often used to calculate forces. Relation (5.3) can be used to determine the motion

only when the relation between the force and the quantity characterizing the motion (time, position vector of the point, velocity, etc.) is known. This relation can either be obtained theoretically, using additional hypotheses which must certainly be verified by experiment, or directly by experimental means.

The relation between force and various physical quantities is obtained from (5.3) in both theoretical and experimental approaches. The relation between the product $m\mathbf{a}$ and the other parameters of motion is established by the observation and study of simple motions. The relations obtained are then generalized to apply to the more complex classes of motion, and the validity of the generalization must be checked experimentally by comparing the conclusions obtained from the equations of motion with the results of experiment. Hence, the general procedure of deriving the law of gravitation from the Kepler laws is typical for determining force as a function of the parameters of motion.

The Coulomb law of force between interacting electric charges, the Biot-Savart law of magnetic intensity, the Weber law of capillary force, the Coulomb friction law of frictional forces between solid bodies, Hooke's law of the relation between the stresses and strains in an elastic body, Newton's law of viscous friction in a fluid, etc. are all determined in an analogous manner.

It is often stated that force can be determined by static means independently of (5.3). Actually, in a number of important cases, in particular when it can be assumed that force depends only on position, the dependence of force on the coordinates can be determined by comparing the required force with the forces known from an analysis of the particular case of motion when $\mathbf{a} = 0$. (In practice, force is often determined by use of the fact that a system of forces is unchanged by a uniform translation. In this case, accelerations do not vary.)

However, in this connection, the following must be kept in mind. First, we use equation (5.3) at $\mathbf{a} = 0$ in comparing the desired force, given by the static definition, with the known forces; second, the statement that relations between forces holding in the static case are also valid in the dynamic case is an additional hypothesis requiring experimental confirmation supplied by (5.3). The experimental check often does not confirm the assumption made. For example, this is the case with frictional forces; it appears that static friction (for a velocity $\mathbf{v} = 0$) and dynamic friction (for $\mathbf{v} \neq 0$) can be different; such is the case with the force of a spring acting on a suspended load. The law relating force to the extension (formula (5.2)), valid for springs of any mass in static measurements, ceases to be correct for motion; the deflections increase with the mass of the spring. If the mass of the spring is small in comparison with the loaded mass, then (5.2) can be considered correct for the motion of the load.

Often it is impossible to define force as a function of time, position, velocity, and acceleration. For example, consider the total force acting on the submerged part of a boat performing a complex loop-shaped motion relative to the water surface. The force acting on the submerged part of the boat depends on the state of motion of water, which is determined by the whole law of motion of the boat. Let the position, velocity, and acceleration at the same instant be identical in two different motions of the boat (the time can be measured from the start of the motion when the boat and water are at rest). It is clearly impossible to say that the forces acting on the submerged part of the boat at this moment will be identical; the forces can differ considerably. The boat could agitate water intensely in the first motion while the motion of water may be calmer at the place considered in the second motion of the boat. Evidently, the forces acting on the boat in this example will depend functionally on the law of motion, i.e. on the whole history of the motion; in other words, the phenomenon will be hereditary in character.

We have thus seen that experimental laws of nature, such as the law of gravitation, Hooke's law, etc., are obtained from an analysis of wide classes of motion in which the magnitude of force is defined as the product of mass by acceleration.

Therefore, in particular problems of kinematics, we may not introduce forces that violate the equation $\mathbf{F} = m\mathbf{a}$, which is backed by sound experimental evidence.

The investigation of mechanical phenomena can be made by similar means if another concept, for example, the kinetic energy of a system, is taken as the fundamental quantity instead of the force. The equation

$$E = \sum \frac{mv^2}{2} \tag{5.6}$$

can be considered the definition of the kinetic energy of a mechanical system. Investigating certain classes of motion of this mechanical system experimentally, we can record the magnitude of the energy E as a function of a number of other mechanical properties. For example, it is established in the motion of a conservative system that the kinetic energy can be represented as a certain function of position and an additive constant h which isolates a known subclass among all the possible motions of the system:

$$E = -V + h \tag{5.7}$$

The quantity V is called the potential energy of a system. Equation (5.6) and condition (5.7) characterizing a conservative system lead to the equation

$$\sum \frac{mv^2}{2} + V = h \tag{5.8}$$

that expresses the law of the conservation of mechanical energy.

At the present time, there is still a number of mechanical phenomena that cannot be investigated by means of (5.3) or (5.7) owing to incomplete knowledge of the behaviour of the forces concerned and the kinetic energy.

In analytic mechanics, it is always understood that the laws of force or an expression for the potential energy are known. The fundamental problems of analytic mechanics are related to the mathematical techniques to be employed, to the methods of integrating the equations of motion, and to the establishment of various equivalent or broader principles which can replace the initial experimental laws. (At the end of the eighteenth century, the main attention and effort of theoretical scientists were directed to the investigation and overcoming of mathematical difficulties (the problems of celestial mechanics, the development of a general theory of differential equations, variational principles, etc.). The initial equations of motion were analysed in general form, giving special attention to the reduction of physical phenomena to mechanical motion and the completeness of mechanics as a science. The fundamental difficulty was to integrate the differential equations of mechanics. As Laplace stated: "Give the initial conditions and this is sufficient to predict the whole future of a motion and to reproduce its whole past." However, it must be noted that it is impossible to regard the theoretical problem of formulating the differential equations of motion as simple, even within the scope of classical mechanics, and, in principle, it is still unsolved. In fact, the problem of formulating the equations of motion, the problem of the effective forces, i.e. of determining the right-hand sides of the differential equations of motion, are the fundamental problems of physical investigations. In many cases, this problem has not been solved even in classical mechanics. In the simplest applications, existing solutions are approximate and are in continued need of improvement.)

The main problem of mechanical or, in general, of physical investigations is to establish the laws expressing forces in terms of the basic characteristic quantities of the state of motion. Further, the significance of these quantities must be explained and the practical value of the laws governing them must be assessed.

One of the Newton's principal achievements is to have shown that the product of mass by acceleration is a quantity which can take on the same value for various bodies and various motions occurring at different places and at different velocities. Further, it is, in general, a quantity which can be determined experimentally as a function of the time, position, and velocity of the points of a system in a number of cases.

However, as we saw, to determine force as a function of the simplest characteristic quantities of motion is not always possible in principle. In these cases, it may be more convenient to replace the

product of mass by acceleration by other quantities and to investigate their behaviour instead.

Let us consider briefly the question of inertial forces. Suppose we have a set of different reference frames moving relative to each other. The acceleration has a different magnitude and direction in various reference frames since their motions are different. The relation between accelerations of a point in reference frames with different relative motions is established in kinematics.

We can establish the relation between force and the basic characteristic quantities of motion experimentally in a certain specific frame of reference, usually one attached to the earth or to the centre of gravity of the solar system.

If we know the laws of force in one reference frame, then we can easily find the product of mass by acceleration, that is, the force, in any other reference frame the motion of which is given relative to the original frame. In this case, of course, we must introduce what is known as the inertial force. For an observer stationary relative to a moving frame of reference, the effective forces are composed of the forces determined in the frame of reference used for experiments (initial reference frame) and of the inertial forces. To a moving observer these have the same mechanical behaviour as any other forces.

§ 6. Nature of the Functional Relations Between Physical Quantities

Physical laws, established either theoretically or directly from experiments, are functional relations between the quantities characterizing the phenomenon under investigation. The numerical values of these dimensional physical quantities depend on the choice of a system of units which has no connection with the substance of the phenomenon. Consequently, the functional relations, which express the physical facts themselves and are independent of the system of units, must have a certain special structure.

We consider the dimensional quantity a defined as a function of the independent dimensional quantities $a_1, a_2, \ldots, a_n$ by

$$a = f(a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n) \tag{6.1}$$

In the process being considered, some of these parameters vary, others are constant.

To be more specific we suppose that the function $f(a_1, a_2, \ldots, a_n)$ represents a certain physical law independent of the choice of the system of units. (Let us stress that functional relation (6.1) is, by hypothesis, just one essential physical relation defining a as a function of the independent quantities $a_1, a_2, \ldots, a_n$. Hence. it is not

the general form of a relation between dimensional quantities which is independent of the choice of the system of units.

For example, the relation

$$a = f(a_1, a_2, \ldots, a_n) + \Phi(a_1, a_2, \ldots, a_n) \ln \frac{P}{mg} \qquad (6.1')$$

is independent of the choice of the system of units, where the function $\Phi(a_1, a_2, \ldots, a_n)$ is arbitrary. Both (6.1′) and (6.1) are satisfied by the quantities $a_1, a_2, \ldots, a_n, g, P$, and m, where g is the acceleration due to gravity, P is the weight, and m is the mass of a certain body. However, (6.1) resolves into two different physical laws: namely (6.1′) and the relation $P = mg$; the latter is superfluous ("parasitic") in this problem. Other artificial examples of this kind could be given.

We shall take our physical law in form (6.1). In what follows we shall only assume that such a relation, which may be many-valued, exists.

Questions concerning the actual theoretical or experimental method of establishing this relation are irrelevant to the argument that follows. Consequently, the investigation of relations between dimensional quantities in implicit form

$$\Phi(a, a_1, a_2, \ldots, a_n) = 0$$

or in the form of several implicit functions for the quantities $a, b, c, \ldots$

$$\Phi_1(a, b, c, \ldots, a_1, a_2, \ldots, a_n) = 0$$
$$\Phi_2(a, b, c, \ldots, a_1, a_2, \ldots, a_n) = 0$$
$$\Phi_3(a, b, c, \ldots, a_1, a_2, \ldots, a_n) = 0$$

is not an analysis of the problem in any more general form. The role of the "parasitic" relations is magnified in such a treatment. The value of dimensional analysis in producing fundamental results in a simple manner is now obscured by complex irrelevant questions, such as the possibility of solving a system of implicit equations.)

Let the first k ($k \leqslant n$) of the dimensional quantities $a_1, a_2, \ldots, a_n$ have independent dimensions (the number of basic units should be larger than or equal to k).

The independence of dimensions means that the dimensions formula of one quantity cannot be represented as a combination, in the form of a power monomial, of the dimensions formulas of the other quantities. For example, the dimensions of length L, velocity L/T, and energy ML^2/T^2 are independent; the dimensions of length L, velocity L/T, and acceleration L/T^2 are dependent. With mechanical quantities, usually not more than three have independent dimensions. We assume that k is the largest number of parameters with independent dimensions, consequently, the dimensions of the quan-

tities a, a_{k+1}, . . ., a_n can be expressed in terms of the dimensions of the parameters a_1, a_2, . . ., a_k.

We take the k independent quantities a_1, a_2, . . ., a_k as the basic quantities and introduce the following notation for their dimensions:

$$[a_1] = A_1, \quad [a_2] = A_2, \ldots, \quad [a_k] = A_k$$

The dimensions of the remaining quantities will be

$$[a] = A_1^{m_1} A_2^{m_2} \ldots A_k^{m_k}$$
$$[a_{k+1}] = A_1^{p_1} A_2^{p_2} \ldots A_k^{p_k}$$
$$\cdots\cdots\cdots\cdots$$
$$[a_n] = A_1^{q_1} A_2^{q_2} \ldots A_k^{q_k}$$

We now alter the units of the quantities a_1, a_2, . . ., a_k by factors of α_1, α_2, . . ., α_k, respectively; the numerical values of these quantities and the quantities a, a_{k+1}, . . ., a_n, in the new system of units, will be given by

$$a_1' = \alpha_1 a_1, \qquad a' = \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_k^{m_k} a$$
$$a_2' = \alpha_2 a_2, \qquad a_{k+1}' = \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_k^{p_k} a_{k+1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_k' = \alpha_k a_k, \qquad a_n' = \alpha_1^{q_1} \alpha_2^{q_2} \ldots \alpha_k^{q_k} a_n$$

respectively. Relation (6.1) becomes, in the new system of units,

$$a' = \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_k^{m_k} a = \alpha_1^{m_1} \alpha_2^{m_2} \ldots \alpha_k^{m_k} f(a_1, a_2, \ldots, a_n)$$
$$= f(\alpha_1 a_1, \ldots, \alpha_k a_k, \alpha_1^{p_1} \alpha_2^{p_2} \ldots \alpha_k^{p_k} a_{k+1}, \ldots, \alpha_1^{q_1} \alpha_2^{q_2} \ldots \alpha_k^{q_k} a_n) \quad (6.2)$$

This relation shows that the function f is homogeneous in the scales α_1, α_2, . . ., α_k. The scales α_1, α_2, . . ., α_k are arbitrary; we choose them so as to cut down the number of arguments of the function f. Let us assume that

$$\alpha_1 = \frac{1}{a_1}, \quad \alpha_2 = \frac{1}{a_2}, \quad \ldots, \quad \alpha_k = \frac{1}{a_k}$$

i.e. select the system of units in such a way that the values of the first k arguments on the right-hand side of (6.2) would equal unity. (For simplicity, we assume that the parameters a_1, a_2, . . ., a_k are finite and different from zero. The final conclusions can be extended to the case when a_1, a_2, . . ., a_k become zero or infinite if the function f is continuous for these values of the arguments.) In other words, recalling that (6.1) (by hypothesis) is independent of the system of units, we establish a system such that the k arguments of the function f would have the fixed constant value, unity.

The numerical values of the parameters $a, a_{k+1}, \ldots, a_n$ are determined in this relative system of units by the formulas

$$\Pi = \frac{a}{a_1^{m_1} a_2^{m_2} \ldots a_k^{m_k}}$$

$$\Pi_1 = \frac{a_{k+1}}{a_1^{p_1} a_2^{p_2} \ldots a_k^{p_k}}$$

$$\cdots\cdots\cdots\cdots$$

$$\Pi_{n-k} = \frac{a_n}{a_1^{q_1} a_2^{q_2} \ldots a_k^{q_k}}$$

where $a, a_1, a_2, \ldots, a_n$ are the numerical values of the quantities considered in the original system of units. It is not difficult to see that the values $\Pi, \Pi_1, \ldots, \Pi_{n-k}$ are independent of the choice of the original system of units since they have zero dimensions relative to the units $A_1, A_2, \ldots, A_k$. It is also evident that the values $\Pi, \Pi_1, \ldots$ $\ldots, \Pi_{n-k}$ are generally independent of the choice of the system of units adopted for the k quantities $a_1, a_2, \ldots, a_k$; therefore, these quantities can be considered dimensionless.

Using the relative system of units, (6.1) can be written as:

$$\Pi = f(1, 1, \ldots, \Pi_1, \ldots, \Pi_{n-k}) \tag{6.3}$$

Hence, the relation between the $n + 1$ dimensional quantities $a, a_1, \ldots, a_n$ reduces to one between the $n + 1 - k$ quantities $\Pi, \Pi_1, \ldots, \Pi_{n-k}$ that are dimensionless combinations of $n + 1$ dimensional quantities. This relation is independent of the choice of units. This general conclusion of dimensions theory is known as the Π-theorem. (This conclusion must be corrected if the function $f(a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_n)$ is discontinuous for zero or infinite values of the first k arguments. The first k arguments in (6.3) can be replaced by unity only at those points at which $a_1, a_2, \ldots, a_k$ are not zero or infinite; consequently, the number of essential arguments on the right-hand side of (6.3) can exceed $n - k$ at such points.)

If it is known that a certain dimensionless quantity is a function of a number of dimensional quantities, then this function can depend only on dimensionless combinations of these dimensional quantities.

By changing the function f, the system of dimensionless parameters in (6.3) can be replaced by another system of dimensionless parameters that are functions of the $n - k$ parameters $\Pi_1, \Pi_2, \ldots, \Pi_{n-k}$.

Clearly, at most $n - k$ independent dimensionless power combinations can be formed from the n parameters $a_1, a_2, \ldots, a_n$ of which not more than k parameters have independent dimensions. This follows directly from (6.3) if we take any dimensionless combination of the quantities $a_1, a_2, \ldots, a_n$ as the quantity a.

Every physical relation between dimensional quantities can be formulated as a relation between dimensionless quantities. This fact is the basic reason why dimensions theory can be applied to the investigation of mechanical problems.

Reducing the number of parameters defining the quantity to be studied restricts the functional relation and simplifies the investigation. In particular, if the number of basic units equals the number of characteristic parameters with independent dimensions, then the relation will be determined completely, to within a constant factor, by using dimensional analysis. In fact, if $n = k$, i.e. all the dimensions are independent, then it is impossible to form a dimensionless combination of the parameters $a_1, a_2, \ldots, a_n$ and, consequently, functional relation (6.3) can be written in the form

$$a = c a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}$$

where c is a dimensionless constant, and the exponents $m_1, m_2, \ldots, m_n$ are easily determined by using the dimensions formula for a. The dimensionless constant c can be determined either by experiment or theoretically by solving the appropriate mathematical problem. Evidently, the more freedom we have in selecting the basic units the greater the use of dimensional analysis.

We saw above that the number of basic units can be selected arbitrarily. However, increasing this number means introducing additional physical constants that must also be included among the characteristic parameters. We increase the number of dimensional constants by increasing the number of basic units; in the general case, the difference $n + 1 - k$, equal to the number of dimensionless parameters in which a physical relation is formulated, remains constant.

An increase in the number of basic units is advantageous only if the entire physical constants introduced are not essential. For example, if we consider a phenomenon in which mechanical and thermal processes occur, then we can introduce two different units, the calorie and the joule, to measure the quantity of heat and mechanical energy, but the dimensional constant J, the mechanical equivalent of heat, must enter into the analysis. Suppose, now, that we are analysing the phenomenon of heat transfer in a moving incompressible ideal fluid: neither the transformation of thermal energy into mechanical energy nor the reverse process occurs in this case and, consequently, the thermal and mechanical processes will proceed independently of the value of the mechanical equivalent of heat. If the value of the mechanical equivalent of heat could be changed, then this in no way would affect the values of the characteristic quantities. Therefore, the constant J does not enter into the physical relation and the increase in the number of basic units permits addi-

tional important information to be obtained by using dimensional analysis.

Later we shall illustrate these conclusions by examples.

§ 7. Parameters Defining a Class of Phenomena

We start every study of mechanical phenomena with a survey, picking out the basic factors that define the quantities of interest to us and, in the broadest sense, drawing on already familiar examples of phenomena to construct a model of the processes under investigation. A sound survey is very often a difficult problem which requires, on the part of an investigator, a great deal of experience, intuition, and a preliminary quantitative explanation of the mechanism of the process being studied. The essence of some problems consists in checking the validity of a plausible hypothesis.

Singling out distinguishing features and developing a real understanding of connections and laws is the basis of the conscious use and control of natural phenomena to solve successfully the multitude of problems presented in the life of mankind.

The properties of matter and the elementary physical laws which play a substantial role in controlling phenomena are characterized by a number of quantities which can be dimensional or dimensionless, variable or constant.

A mechanical system and the state of its motion are determined by a number of dimensional and dimensionless parameters and functions.

Suppose that several different mechanical systems perform a certain motion; then we can always restrict the number of admissible systems and motions to those which can be defined by a finite number of dimensional and dimensionless parameters. The limitation of the class of admissible systems and motions can always be attained by imposing additional conditions on the abstract parameters and the type of function associated with the problem in dimensionless form.

Dimensional analysis enables us to draw conclusions by using arbitrary or special systems of units to describe physical laws. Consequently, when listing the parameters defining a class of motions, it is necessary to include all the dimensional parameters related to the substance of the phenomena independently of whether these parameters are constant (in particular, they can be physical constants) or can vary for different motions of the class isolated. It is important that dimensional parameters should be able to assume various numerical values in different systems of units, although they are possibly identical for all the motions being considered. For example, when considering the motion in which the weight of a body is of importance, we must certainly take into account the acceleration due to gravity g as a physical dimensional constant, although the

value of g is constant under all actual conditions. After the acceleration due to gravity g has been introduced as a characteristic parameter, we can, without introducing complications, extend the class of motions artificially by introducing those in which the acceleration due to gravity g assumes different values. Such a method permits qualitative conclusions of practical value to be obtained in a number of cases.

We now consider how to find the system of parameters defining a class of phenomena.

If the problem is formulated mathematically, a table of the parameters defining the phenomenon is always easily extracted. To do this, we need to note all the dimensional and dimensionless quantities that are required to determine the numerical values of all unknowns from the equations of the problem. In some cases, a table of the characteristic parameters can be formed without writing down the equations. It is then possible to single out the factors needed to find the required quantity; sometimes its numerical values can only be found experimentally.

It is necessary, when compiling the system of characteristic parameters, to form a clear picture of a phenomenon, just as when formulating the equations of a problem.

However, less need be known when using dimensional analysis than when formulating the equations of motion of a mechanical system. Several equations of motion can exist for the same system of characteristic parameters. The equations of motion show not only which parameters the required quantities depend on, but also contain all the functional relations determined by the mathematical form of a problem.

These arguments show that dimensional analysis is limited in its scope. We cannot determine functional relations between dimensionless quantities by use of dimensional analysis alone.

The conclusions of dimensional analysis cannot be changed if we multiply the various terms in the equations of motion by positive or negative dimensionless numbers or functions depending on the system of characteristic parameters; yet modifications to the equations of this sort can substantially influence the character of physical laws. (For example, a change in the sign of certain forces in the equations of motion of the system in question can have an important effect on the laws of motion; but all conclusions of dimensional analysis remain invariant under this operation.)

Every system of equations which includes a mathematical description of a controlling phenomenon can be formulated as a relation between dimensionless quantities. None of the conclusions of dimensional analysis are changed by a variation of physical laws when they are represented in the form of the relations between identical dimensionless quantities.

The system of characteristic parameters must be complete. Some of the characteristic parameters, including dimensional physical constants, must have dimensions in terms of which the dimensions of all the dependent parameters can be expressed. (If the system of characteristic parameters is incomplete and it is impracticable to extend it, then the defining quantity equals either zero or infinity. We often encounter such cases when assigning initial conditions of the "source type" by using δ-function.) Certain characteristic parameters may be physical dimensional constants. As an example of this requirement, let us consider the parameters which can determine the static state of a gas. It is wrong to assert that the state of a gas is determined only by the two dimensional quantities, the absolute temperature T ($[T] = \mathrm{C}^\circ$) and density ρ ($[\rho] = \mathrm{M}/\mathrm{L}^3$), because the pressure p is finite, nonzero, and has dimensions independent of those of the temperature and the density.

Now let us assume that the state of the gas is determined by the values of the temperature, the density, and a physical constant, say the coefficient of specific heat c_V', measured in mechanical units ($[c_V'] = \mathrm{L}^2/T^2\,\mathrm{C}^\circ$). Denoting the mechanical equivalent of heat by J kgf-m/cal, we shall have

$$c_V' = Jc_V$$

where c_V is the specific heat in thermal units ($[c_V] = \mathrm{cal/mass}\cdot\mathrm{C}^\circ$). The dimensions of pressure can be expressed in terms of the dimensions of T, ρ, and c_V'; consequently, the assumption made is admissible from the point of view of dimensional analysis. Since the dimensions of T, ρ, and c_V' are independent, then from the assumption that

$$p = f(T, \rho, c_V')$$

we at once deduce the Clapeyron equation

$$\frac{p}{c_V'\rho T} = c \quad \text{or} \quad p = \rho RT$$

where c is a dimensionless constant, and R denotes the dimensional constant $cc_V' = cJc_V$.

Hence, the Clapeyron equation can be considered as a consequence of the single hypothesis that pressure, density, temperature, and specific heat are connected, *independently of the values of the other characteristics*, by a relation that has a physical meaning. Examples discussed in later chapters illustrate methods of combining dimensional analysis with reasoning resulting from symmetry, from linearity of the problem, from the mathematical properties of the functions for small or large values of the characteristic parameters, and so on.

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The unit of mass is retained as the mass of the international prototype of kilogram: the bar made of platinum-iridium alloy kept in the French Bureau of Weights and Measures (Ist GCWM, 1889, and IIIrd GCWM, 1901).
XIIIth GCWM, 1967, defined the second as 9,192,631,770 periods of radiation emitted in the transition between the two hyperfine-splitting levels of the ground state of the cesium-133 atom.
3. The ampere is defined as the intensity of a constant direct current which, flowing through two parallel straight infinitely long conductors with circular cross section of an infinitesimal area, spaced in vacuum by the metre, would produce on each one-metre-long segment of the conductors the force of interaction equal to 2×10^{-7} newton (ICWM, 1946; IXth GCWM, 1948). The newton is the unit of mechanical force in the SI; 1 kg = 9.80665 N.
The kelvin is equal to 1/273.16 of the absolute temperature of the triple point of water (XIIIth GCWM, 1967).
The candela is defined as the luminous intensity, in the perpendicular direction, of a surface of 1/600,000 square metre of a black body emitter at the temperature of freezing platinum under a pressure of 101,325 pascals (Pa) (XIIIth GCWM, 1967). One mm Hg equals 133.322 Pa, and 1 kgf/mm^2 = = 9.80665 Pa.
The mole is equal to the amount of substance in the system comprising the same number of structural elements as that comprised in 0.012 kg of carbon-12. When this unit is applied, the structural elements must be specified; they may be atoms, molecules, ions, electrons, or other particles, or specified groups of particles (XIVth GCWM, 1971).
4. According to the original idea of the commission of the French Academy of Sciences that established the metric system, the metre was to be defined as 0.25×10^7 of the length of the meridian through Paris, and the kilogram was to be the mass of a cubic decimetre of distilled water at 4 °C and at atmospheric pressure. Uncertainties in the concept of the meridian length and unavoidable errors in measuring it made a new definition of unit length (see [2]) more practicable. The mass of the reference bar of platinum-iridium alloy (see [2]), originally equal to that of the cubic decimetre of distilled water at 4 °C and at atmospheric pressure, is now taken as the standard of mass.

CHAPTER II

SIMILARITY, MODELLING, AND VARIOUS EXAMPLES OF THE APPLICATION OF DIMENSIONAL ANALYSIS

§ 1. Motion of a Simple Pendulum

As the first example, we shall consider the classical problem of motion of a simple pendulum.

A simple pendulum (Fig. 1) is a heavy particle suspended by a weightless inextensible string which is fixed at its other end. We shall assume that the motion of the pendulum is two-dimensional.

We introduce the following notation: l is the length of the pendulum, φ is the angle between the string and the vertical, t is time, m is the mass of the particle, and N is the tension in the string. If the resistance forces are neglected, then the problem reduces to the solution of the equations

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\sin\varphi \tag{1.1}$$

$$m\left(\frac{d\varphi}{dt}\right)^2 l = N - mg\cos\varphi \tag{1.2}$$

with the initial conditions

$$\varphi = \varphi_0 \quad \text{and} \quad \frac{d\varphi}{dt} = 0$$

when $t = 0$, i.e. the initial instant is taken as that at which the pendulum is deflected by the angle φ_0 and the velocity is zero.

It is evident from equations (1.1), (1.2), and the initial conditions that we may choose as our system of characteristic parameters the following quantities:

$$t,\ l,\ g,\ m,\ \varphi_0$$

The numerical values of all the remaining quantities are determined completely by the values of these parameters. Therefore, we can write

$$\varphi = \varphi(t,\ \varphi_0,\ l,\ g,\ m), \qquad N = mgf(t,\ \varphi_0,\ l,\ g,\ m) \tag{1.3}$$

where φ and f are dimensionless functions.

The numerical values of the functions φ and f should not depend on the system of units. The form of these functions can be determined either by solving equations (1.1) and (1.2) or by experiment.

From these general considerations, it follows that the five dimensional arguments of the functions φ and f can be reduced to only two. These are dimensionless combinations of t, l, g, m, and φ_0 since there are three independent units.

The two independent dimensionless combinations

$$\varphi_0 \quad \text{and} \quad t\sqrt{\frac{g}{l}} \tag{1.4}$$

can be formed from the quantities t, l, g, m, and φ_0. All the other dimensionless combinations formed from t, l, g, m, and φ_0 or, generally, from any quantities determined by these parameters will be the functions of combinations (1.4). Therefore, we can write

$$\varphi = \varphi\left(\varphi_0, t\sqrt{\frac{g}{l}}\right) \tag{1.5$'$}$$

$$N = mgf\left(\varphi_0, t\sqrt{\frac{g}{l}}\right) \tag{1.5$''$}$$

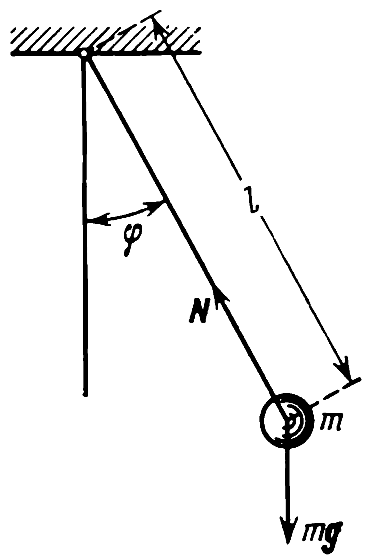

Fig. 1. Simple pendulum

Formulas (1.5), obtained by dimensional methods, show that the law of motion is independent of the mass of the particle while the tension of the string is directly proportional to the mass. These conclusions also follow from equations (1.1) and (1.2). The quantity $t\sqrt{g/l}$ can be considered the time expressed in a special system of units in which the length of the pendulum and the acceleration due to gravity are taken equal to unity.

Let us denote any characteristic time interval, for example, the time taken by the pendulum to move between the extreme and vertical positions or between two identical phases, i.e. the period of oscillation, by $\mathcal{T}$ (the existence of a periodic motion can be taken as a hypothesis or as a result known from additional data). We shall have

$$\mathcal{T} = f_1(\varphi_0, l, g, m) = \sqrt{\frac{l}{g}}\, f_2(\varphi_0, l, g, m)$$

The function f_2 is a dimensionless quantity and, since a dimensionless combination cannot possibly be formed from l, g, and m, it is clear that the function f_2 is independent of l, g, and m. Therefore,

$$\mathcal{T} = \sqrt{\frac{l}{g}}\, f_2(\varphi_0) \tag{1.6}$$

Formula (1.6) establishes the relation between the time $\mathcal{T}$ and the length of the pendulum. It is impossible to determine the form of the function $f_2(\varphi_0)$ by dimensional analysis; this must be found either theoretically from equation (1.1) or experimentally.

Formula (1.6) can be obtained directly from relation (1.5′). In fact, (1.5′) gives, for the period of oscillation,

$$\varphi_0 = \varphi\left(\varphi_0, \mathcal{T}\sqrt{\frac{g}{l}}\right)$$

Solving this equation, we obtain formula (1.6).

If $\mathcal{T}$ is the period of oscillation, then it is evident, from symmetry considerations, that the value of the period $\mathcal{T}$ is independent of the sign of φ_0, that is,

$$f_2(\varphi_0) = f_2(-\varphi_0)$$

Therefore, the function f_2 is an even function of the argument φ_0. Assuming that the function $f_2(\varphi_0)$ is regular for small φ_0, we can write

$$f_2(\varphi_0) = c_1 + c_2\varphi_0^2 + c_3\varphi_0^4 + \ldots \tag{1.7}$$

The terms in φ_0^2 and higher-order terms can be discarded for small oscillations and we obtain the following formula for the period $\mathcal{T}$:

$$\mathcal{T} = c_1\sqrt{\frac{l}{g}} \tag{1.8}$$

The solution of equation (1.1) shows that $c_1 = 2\pi$. Hence, we see that the formula for the period of oscillation of the pendulum can be obtained by dimensional analysis to within the accuracy of a constant factor when the amplitude is small.

Formulas (1.5) and (1.6) still remain valid if we take

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l} f(\varphi)$$

where $f(\varphi)$ is any function of φ, instead of (1.1). In general, the validity of formulas (1.5) and (1.6) results from the single condition that the state of motion is determined by the following parameters:

$$t, \ l, \ g, \ m, \ \varphi_0$$

In order to establish this system of parameters, we had to work with the equations of motion; but it can be derived independently of them. We must choose l and m for the pendulum characteristics. Furthermore, g must be involved since the nature of the phenomenon is determined by gravity. Finally, φ_0 and t must appear since the actual motion is determined by the maximum angle of deflection φ_0 and by the time t.

§ 2. Flow of a Heavy Fluid Through a Spillway

We next consider the problem of flow of a heavy fluid through a spillway (Fig. 2) consisting of a vertical wall with a triangular orifice which is symmetrical about the vertical and includes an angle α

equal to 90°. The fluid flows under a head h equal to the height of the fluid level above the vertex of the triangle at a large distance from the spillway exit. We assume, for simplicity, that the vessel holding the fluid is very large and, consequently, the fluid motion can be considered steady.

The properties of inertia and weight, characterized by the density ρ and by the acceleration of gravity g, have special significance in the streaming motion of a fluid.

A steady flow of a liquid through the spillway is defined completely by the parameters

$$\rho,\ g,\ h$$

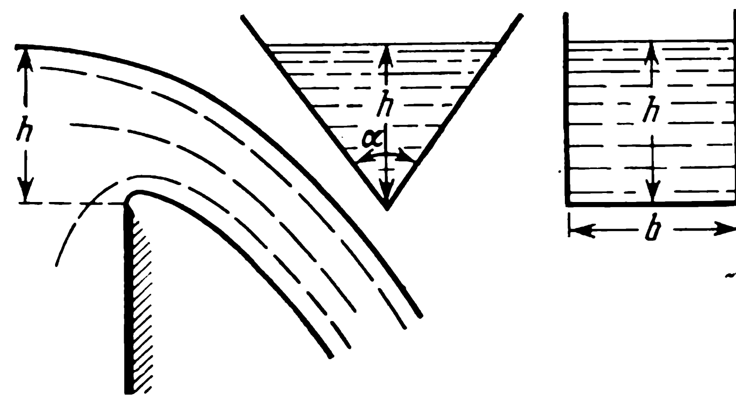

Fig. 2. Flow of a heavy liquid through a spillway.

The weight Q of the liquid that flows through the spillway opening per unit time can be a function only of these parameters, so that

$$Q = f(\rho,\ g,\ h)$$

Using dimensional analysis, it is not difficult to find the form of this function. Actually, the dimensions of Q are kgf/s. The combination $\rho g h^3 \sqrt{g/h}$ also has the dimensions of kgf/s. Consequently, the ratio

$$\frac{Q}{\rho g^{3/2} h^{5/2}}$$

is dimensionless. This ratio is a function of the quantities ρ, g, and h from which a dimensionless combination cannot possibly be formed. Consequently, we can write

$$\frac{Q}{\rho g^{3/2} h^{5/2}} = C$$

or

$$Q = C \rho g^{3/2} h^{5/2} \tag{2.1}$$

where C is an absolute constant which is determined, most readily, by experiment. This formula determines the relation between the mass flow, the head h, and the density ρ completely.

We can extend this analysis to cover spillways with different angles α. In this case, the angle α is added to the system of characteristic parameters and (2.1) becomes

$$Q = C(\alpha)\, \rho g^{3/2} h^{5/2} \tag{2.2}$$

that is, the constant C will depend on the angle α.

If the spillway has a rectangular shape with width b, then the system of characteristic parameters will be

$$\rho,\ g,\ h,\ b$$

All the dimensionless quantities are determined by the parameter h/b. In this case, we replace (2.1) by

$$Q = f\left(\frac{h}{b}\right)\rho g^{3/2}h^{5/2} \tag{2.3}$$

The function $f\ (h/b)$ can be determined experimentally by observing the flow through spillways of various widths b but with constant h. Formula (2.3) can be applied to cases in which the width b is constant but the head h varies, i.e. to cases for which no experimental results are available.

This example shows the usefulness of dimensional analysis in cutting down the number of experiments with a consequent saving in time as well as cost of apparatus. In experiments the particular quantities to be investigated can be replaced by another quantities. Complete information about flow of oil, mercury, etc. can be derived from experiments on water.

§ 3. Fluid Motion in Pipes

The value of dimensional analysis and similarity theory was first demonstrated with special clarity in hydraulics during the study of fluid motion in pipes. Despite the practical importance and simplicity of the reasoning of dimensional analysis, its use in hydraulics problems, leading to a considerable advance in the subject, only occurred at the end of the nineteenth century, following the works of Osborne Reynolds [1].

Empirical formulas, proposed by various authors, have long been used in hydraulics. These formulas contained a number of dimensional constants the values of which were determined by special experimental conditions and by fluid properties.

The reasoning of dimensional analysis, besides giving a clearer and more general formulation of the problem, led to the empirical laws governing the motion of fluids at different temperatures in pipes of different diameters and at different velocities.

We now describe and formulate our problem. We consider a cylindrical pipe of uniform cross section (Fig. 3). The geometry of the pipe is then defined completely by the cross-sectional area or by a characteristic length a. The radius or diameter is usually taken as the characteristic length for circular pipes. The pipe length is assumed to be sufficiently large to justify neglect of end effects, and we can therefore assume that the pipes are infinitely long.

We assume that the motion is steady.

We neglect compressibility but take account of inertia and viscosity represented by the density ρ and the coefficient of viscosity μ. Since the coefficient of viscosity depends on the temperature, we can allow for temperature effects by varying the viscosity. (Here, however, we assume that the temperature is constant throughout the whole fluid.)

In order to determine the motion of the fluid, we need to know either the pressure drop along the pipe, or the fluid discharge per unit

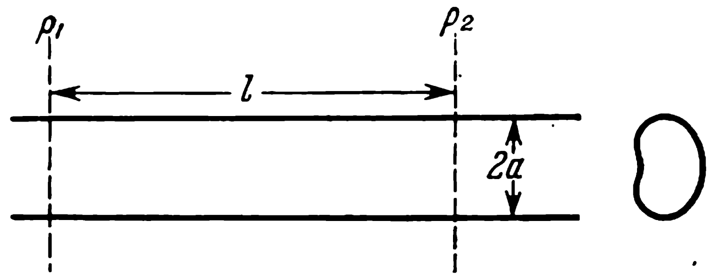

Fig. 3. Motion of an incompressible fluid in a cylindrical pipe.

time through the pipe cross section, or the average fluid velocity $\bar{u}$ over the pipe cross section, or other equivalent data.

Therefore, the complete motion in the pipe is determined by the system of parameters

$$\rho, \ \mu, \ a, \ \bar{u}$$

All mechanical characteristics of motion are functions of these parameters.

For example, consider the pressure drop along the pipe. The pressure drop per unit length of the pipe is given by the quantity

$$\frac{p_1 - p_2}{l}$$

where p_1 and p_2 are the pressures in cross sections of the pipe a distance l apart.

The combination

$$\frac{p_1 - p_2}{l \dfrac{\rho \bar{u}^2}{2a}} = \psi$$

is a dimensionless quantity and is called the resistance (drag) coefficient of the pipe.

The resistance of a section of the pipe of length l is

$$P = (p_1 - p_2) S = \psi \frac{l}{a} S \frac{\rho \bar{u}^2}{2} \tag{3.1}$$

where S is the area of the pipe cross section.

Only one independent dimensionless combination

$$\frac{\bar{u}a\rho}{\mu} = \mathrm{Re}$$

called the Reynolds number can be formed from the four characteristic parameters ρ, μ, a, and $\bar{u}$. All dimensionless quantities depending on these four parameters are functions of the Reynolds number. In particular,

$$\psi = \psi(\mathrm{Re}) \tag{3.2}$$

The problems of determining the pipe resistance or the fluid discharge as a function of the pressure drop reduce to finding the function $\psi(\mathrm{Re})$. This function can be found experimentally by measuring the variation of the resistance with the velocity (or with the mass flow

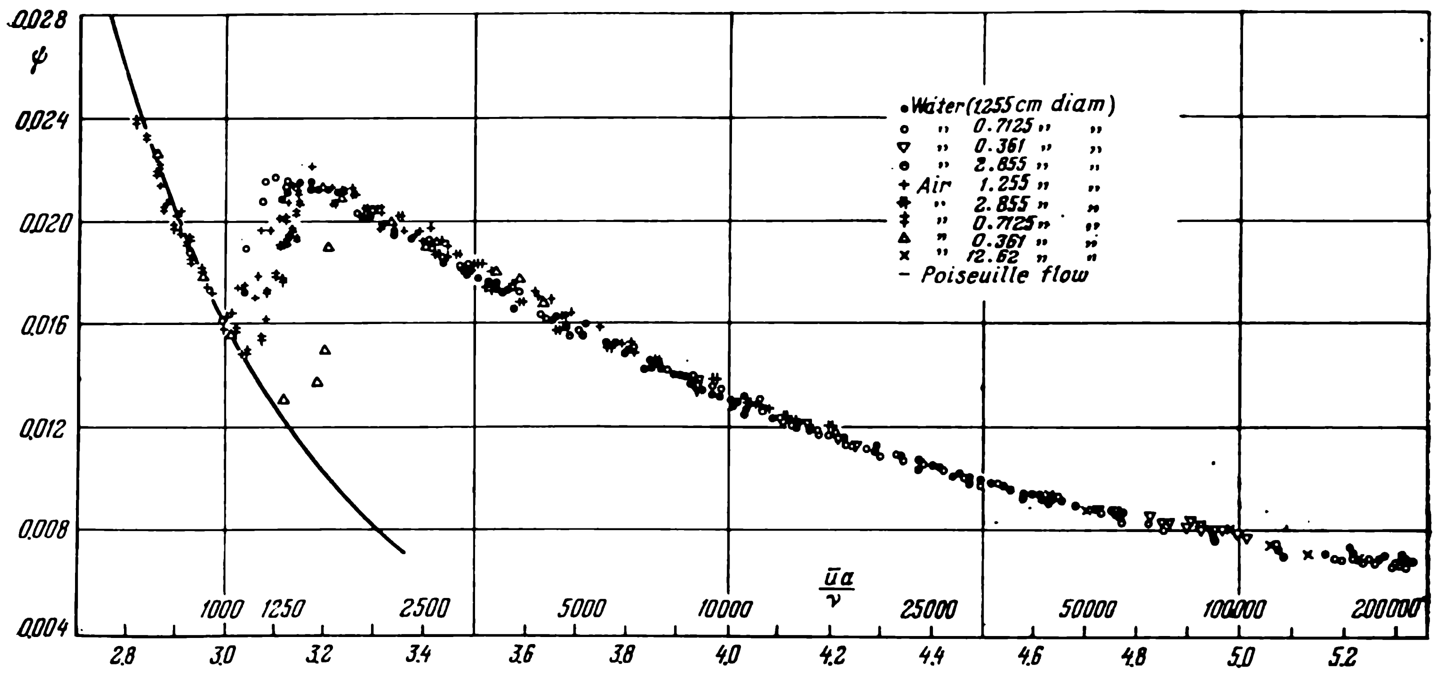

Fig. 4. Experimental data on the resistance coefficient of circular cylindrical pipes. (Measurements of Stanton and Pannel; a is the pipe radius, $\nu = \mu/\rho$; the lower scale corresponds to log Re).

of the stream) when water moves along a pipe. The results obtained can be used to analyse the motion of other fluids in pipes of different diameters. For example, the properties of motion of air in pipes, etc. (Fig. 4) can be deduced from experimental data on water under certain conditions (when compressibility can be neglected, i.e. at speeds considerably less than the speed of sound).

Experiment shows that the fluid motion in pipes is of two very different types, namely, laminar motion and turbulent motion. In laminar motion, the fluid particles in a cylindrical pipe move in lines parallel to the generators of the pipe; in turbulent motion, there is a disordered mixing of the fluid in a direction perpendicular

to the generators. A turbulent flow can be considered steady motion only on the average.

In some cases, laminar motion of fluids in a pipe is weakly stable, or even unstable, and degenerates into turbulent motion.

The stability property is characteristic of fluid motion as a whole and, consequently, must be defined by the Reynolds number Re for smooth pipes. Experiment is in good agreement with this conclusion. Laminar motion is stable for low values of the Reynolds number and unstable for large values. The Reynolds number determines the mode of motion. The boundary of stability of laminar motion is defined by a certain value of the Reynolds number known as the critical value. The critical Reynolds number for circular pipes is of the order of $\mathrm{Re}_{cr} = 1000\text{-}1300$.

The laminar mode is found in fluids of high viscosity moving at low velocities in pipes of small diameter (for example, in capillary tubes). The turbulent mode is found in fluids of low viscosity moving at high velocities in pipes of large diameter.

Experimental data show that the function $\psi(\mathrm{Re})$ has two branches, one of which corresponds to the laminar and the other to the turbulent mode of motion. A transition region is observed in the vicinity of the critical value of the Reynolds number.

All fluid particles in laminar motion in a cylindrical pipe move in straight lines parallel to the pipe axis at a constant speed, i.e. at zero acceleration. This fluid motion in pipes is called the Hagen-Poiseuille flow. The inertial property of a fluid represented by the parameter ρ can only be felt if the acceleration is nonzero (as we know, the density and acceleration only enter into the derivatives in the equations of motion). Consequently, the resistance must be independent of ρ in laminar motion. Therefore, the right-hand side of (3.1) must be independent of ρ in laminar motion, and the function ψ (Re) must be

$$\psi = \frac{C}{\mathrm{Re}} = \frac{C\mu}{\rho a \bar{u}} \tag{3.3}$$

where C is a dimensionless constant determined by the geometric shape of the pipe cross section, and a is the pipe linear dimension. For a circular pipe, C is easily calculated theoretically: $C = 16$.

The pipe resistance in the case of laminar motion is then given by

$$P = \frac{1}{2}\frac{S}{a^2} C\mu l\bar{u} = C_1 \mu l \bar{u} \tag{3.4}$$

where C_1 is a dimensionless constant depending on the shape of the pipe cross section. Formula (3.4) is easily obtained directly if the three quantities a, μ, and $\bar{u}$ are taken as the only characteristic parameters and if account is taken of the fact that P is proportional to l.

If the pressure drop under which the fluid moves is given, then it is convenient to take the characteristic parameters:

$$\rho,\ \mu,\ a,\ \text{and}\quad i=\frac{p_1-p_2}{l}$$

In this case, the flow mode is determined by the dimensionless parameter

$$\frac{\rho i a^3}{\mu^2}=J$$

It is easy to see, from (3.1), that

$$J=\frac{1}{2}\,\mathrm{Re}^2\,\psi(\mathrm{Re}) \tag{3.5}$$

This relation gives the variation of J with Re in terms of the function ψ (Re). Let us denote the volume of fluid flowing through the pipe cross section per unit time (the volume discharge of the pipe) by

$$Q=\bar{u}S$$

The dimensionless combination

$$\frac{Q\rho}{\mu a}=\mathrm{Re}\,\frac{S}{a^2}$$

is a function of J, i.e.

$$Q=\frac{\mu a}{\rho}\,f(J) \tag{3.6}$$

The form of the function $f(J)$ is easily determined for laminar motion. From (3.3) and (3.5), we find

$$Q=\frac{2}{C}\,\frac{S}{a^2}\,\frac{ia^4}{\mu}=C_2\,\frac{ia^4}{\mu} \tag{3.7}$$

where the dimensionless constant C_2 depends on the shape of the pipe cross section. For a circular pipe,

$$C_2=\frac{\pi}{8}$$

Formula (3.7) is Poiseuille's law which was established experimentally by Hagen in 1839 and by Poiseuille in 1840. The very good agreement of this law with experiment is one of the main confirmations of the validity of the law of viscous friction in fluids and of the initial survey of the phenomenon.

§ 4. Motion of a Body in a Fluid

A survey of the problem of the motion of an airplane, submarine etc. leads to the problem of the forward uniform motion of a solid body in an infinite fluid.

If we regard the geometric shape of the surface of the body as fixed, then the surface of the body will be completely specified by a characteristic length d. We consider the forward motion of the body parallel to a fixed plane. Let us denote the velocity of the motion and the angle which defines the velocity direction (Fig. 5) by v and α, respectively; the quantities v and α can be different. Further, we assume that the fluid is incompressible, but take account of the inertia and viscosity of the fluid. For simplicity, we assume that body forces are absent. The pressure distribution over the body surface and the total forces exerted by the fluid on the body depend on the state of the disturbed fluid motion.

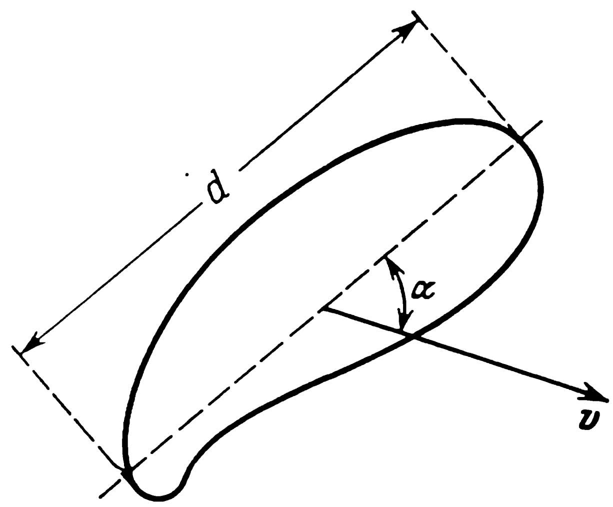

Fig. 5. Motion of a solid body in a fluid.

The steady-state motion of a fluid is defined for a body of a given shape by a system of the five parameters:

$$d,\ v,\ \alpha,\ \rho,\ \mu$$

(The pressure at infinity, p_0, which can be assigned arbitrarily, is not introduced into this system of parameters for the following reasons. The fluid is incompressible, consequently, a change in p_0 cannot influence the velocity field. The pressure difference $p - p_0$ can always be considered instead of the value of the total pressure p. Hence, it is evident that the quantity p_0 is not essential and, consequently, it is not necessary to introduce it as a characteristic parameter. However, when the fluid motion involves cavitation phenomena resulting from vaporization in the lower pressure regions, it is necessary to include the quantity $p_0 - p'$ among the characteristic parameters, where p' is the vapour pressure at a given temperature. The quantity p_0 or an equivalent parameter must be included as a characteristic parameter for a compressible fluid.

The dimensionless parameter $\varkappa = 2\,\frac{p_0 - p'}{\rho v^2}$ is also important in motions accompanied by cavitation. When studying the effect of the cavitation number $\varkappa$ in experiments, its value can be varied either by changing p_0, v or, artificially, p'. Different fluids can also be used and the density ρ can be varied.)

All the dimensionless mechanical quantities related to the motion can be considered as functions of $5 - 3 = 2$ dimensionless parameters: the angle of attack α and the Reynolds number

$$\frac{v\,d\rho}{\mu} = \mathrm{Re}$$

Let us denote the force exerted by a fluid on a body by W (in the subsequent discussion, it does not matter whether we understand W to be the total resistance (drag) or one of its components, that is, the drag directed opposite to the fluid motion or the lift normal to the velocity direction). It follows from the general theorem of dimensional analysis that the dimensionless quantity $W/(\rho d^2v^2)$ is a function of the angle of attack α and the Reynolds number Re. Consequently,

$$W = \rho d^2 v^2 f(\alpha, \mathrm{Re}) \tag{4.1}$$

The determination of the function $f(\alpha, \mathrm{Re})$ is the most important problem of theoretical and experimental aerodynamics and hydrodynamics.

Evidently, the effect of viscosity on the motion is felt only through that of the Reynolds number with the system of parameters chosen.

When the velocity or dimensions of a body increase, certain general conclusions can be made about the role of the fluid viscosity from the form of the formula $\mathrm{Re} = v\,d\rho/\mu$. For example, the Reynolds number increases as the velocity or the linear dimensions of the body increase. But the Reynolds number must remain constant in order to preserve the effect of viscosity, since any change in the Reynolds number can be referred to a change in the coefficient of viscosity; if the product $v\,d\rho$ increases, then the coefficient of viscosity μ must be increased in order to keep the Reynolds number constant. Therefore, the motion of honey (large μ/ρ), caused by the motion of a large body, is similar to the motion of water (small μ/ρ), caused by the motion of a small body, at identical speeds. Or the motion of a body in honey at a high speed is the same as the motion of the same body in water at a low speed. The similarity of the motions is expressed by the fact that all the dimensionless quantities are identical for these motions.

Furthermore, these considerations show that the effect of viscosity on a body moving in the same fluid decreases as the velocity and dimensions of the body increase. (We assume in this case that, other conditions being equal, the viscous effect is reduced as the coefficient of viscosity decreases.) Theoretical investigations and experimental data show that the effect of the fluid viscosity decreases for high values of the Reynolds number and becomes unimportant in certain cases. Neglecting the viscosity, i.e. putting $\mu = 0$, we arrive at the concept of an ideal fluid.

The number of characteristic parameters in problems concerning the motion of a body in an ideal fluid is cut down to four:

$$d, \ \alpha, \ \rho, \ v$$

All the dimensionless characteristics in an ideal fluid are determined by the angle of attack (incidence) α; consequently, formula

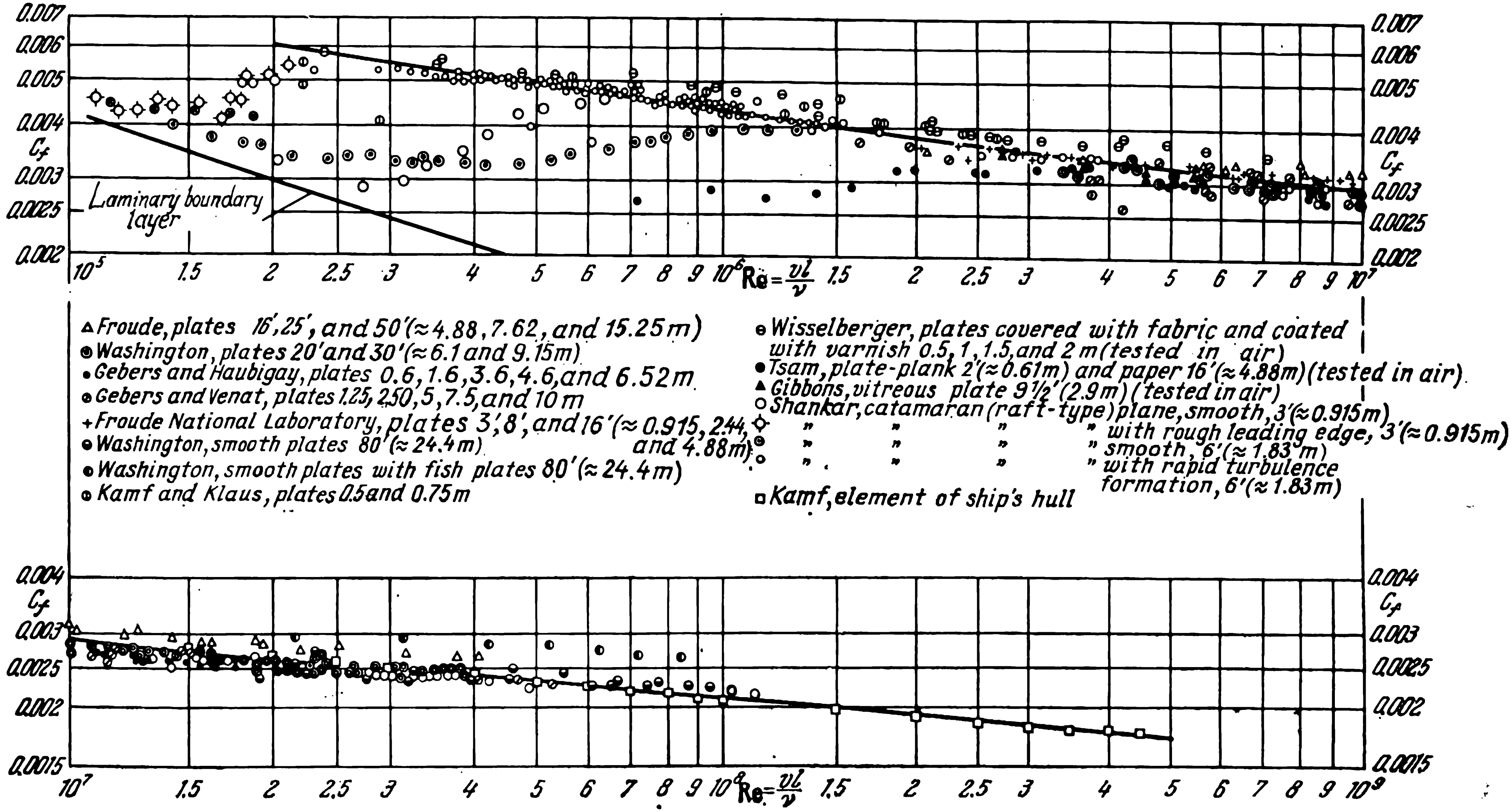

Fig. 6. The drag coefficient $C_f = \dfrac{W}{lb\rho v^2/2}$ of plane square plates towed parallel to their planes (l is the dimension in the direction parallel to the velocity, and b is the plate width).

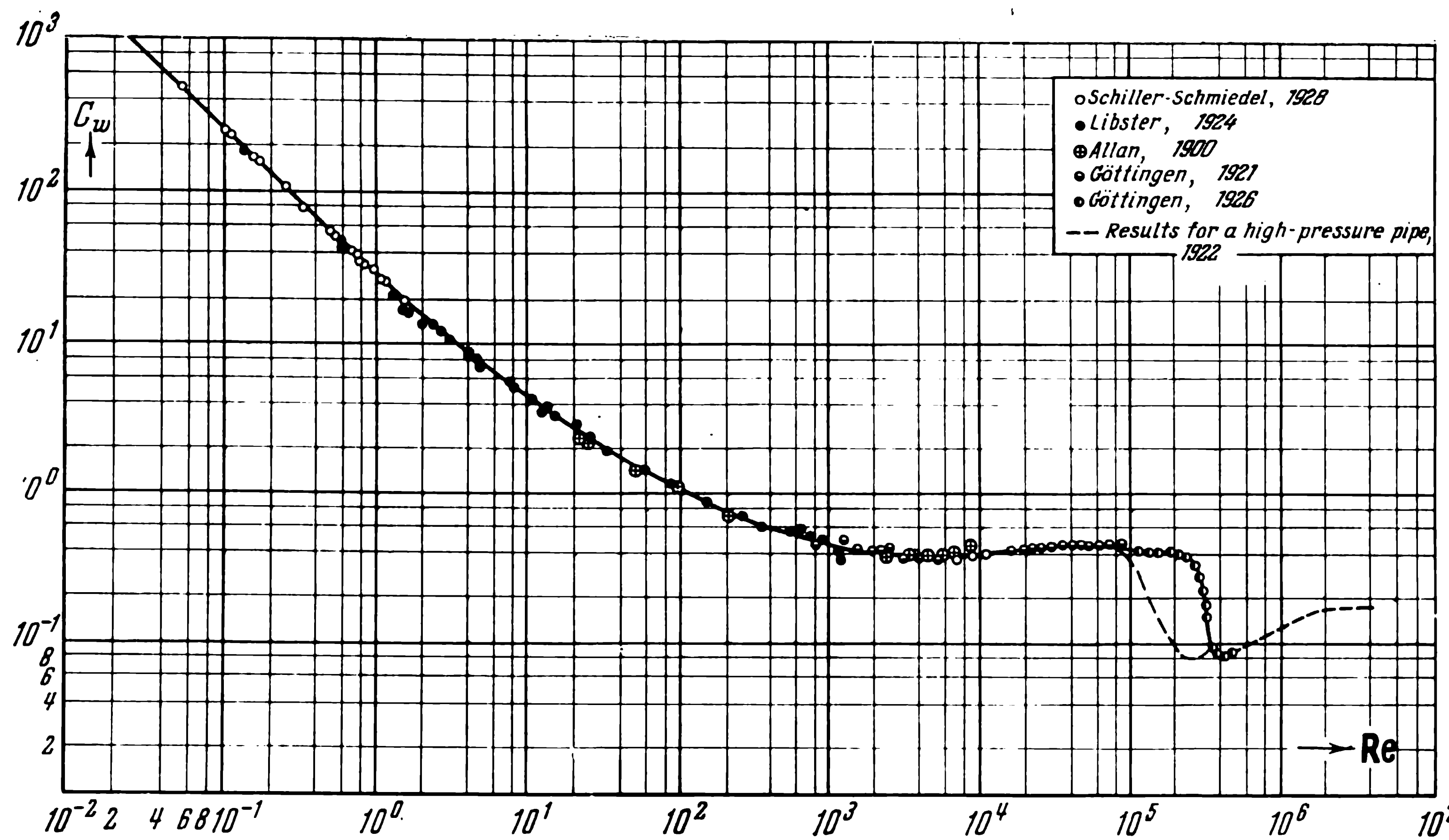

Fig. 7. The drag coefficient $C_w = \dfrac{W}{(\pi d^2/4)(\rho v^2/2)}$ of a sphere as a function of the Reynolds number $\mathsf{Re} = vd/\nu$ (d is the diameter of the sphere).

(4.1) is replaced by

$$W = \rho d^2 v^2 f_1(\alpha) \tag{4.2}$$

Therefore, the forces acting on a body moving in an ideal incompressible fluid are proportional to the velocity squared. This law is approximately correct for a viscous fluid for high enough values of the Reynolds number.

The functions $f(\alpha, \mathrm{Re})$ and $f_1(\alpha)$ in (4.1) and (4.2) considerably depend, for bodies of different shapes, on abstract parameters determined by the body shape as well as on the angle of attack. Figures 6 and 7 show experimental data on the variation of the drag coefficient with the Reynolds number. Figure 8 shows the nature of the effect of the angle of attack on the drag and lift of a wing.

Now let us consider the case of very slow motion corresponding to low values of the Reynolds number.

As the Reynolds number decreases, the role of the viscous forces increases. If we neglect the inertial forces in comparison with the viscous forces, then this is equivalent to assuming that the parameter ρ is unimportant. There are the four characteristic parameters in this case:

$$d, \ \alpha, \ v, \ \mu$$

consequently, all the dimensionless characteristics will also depend only on the angle of attack α. Therefore,

$$W = \mu \, dv \, f_2(\alpha) \tag{4.3}$$

Hence, it is evident that the drag and the lift are proportional to the velocity, the coefficient of viscosity, and the linear scale d. This law, called Stokes' law, is in good agreement with experiment on small bodies at low speeds; for example, in the precipitation of solid particles in a fluid.

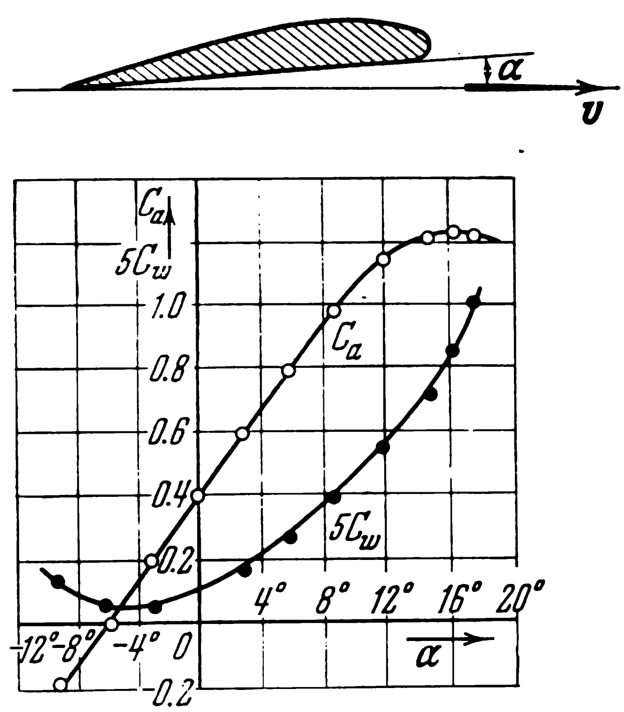

Fig. 8. Typical curves of the litf coefficient $C_a = 2A/(\rho S v^2)$ and the drag coefficient $C_w = 2W/(\rho S v^2)$ as a function of the angle of attack α for a wing (S is the wing planform area).

The function $f_2(\alpha) = \text{const} = c$ for a sphere, i.e. f_2 is independent of the angle α. The theoretical value of the coefficient c for slow motions of a sphere under the above assumptions (which reduce to neglecting the inertia terms in the Navier-Stokes equations) was calculated by Stokes; it was found that $c = 3\pi$ (if d is the diameter of the sphere).

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We see that dimensional analysis permits the form of the function $f(\alpha, \mathrm{Re})$ to be determined for very small and very large values of the Reynolds number. When $\mathrm{Re} \to \infty$, we arrive at an ideal fluid; in this case, the function $f(\alpha, \mathrm{Re})$ tends to a certain function $f_1(\alpha)$ that is independent of the Reynolds number. At low values of the Reynolds number, the effect of the fluid viscosity is paramount and

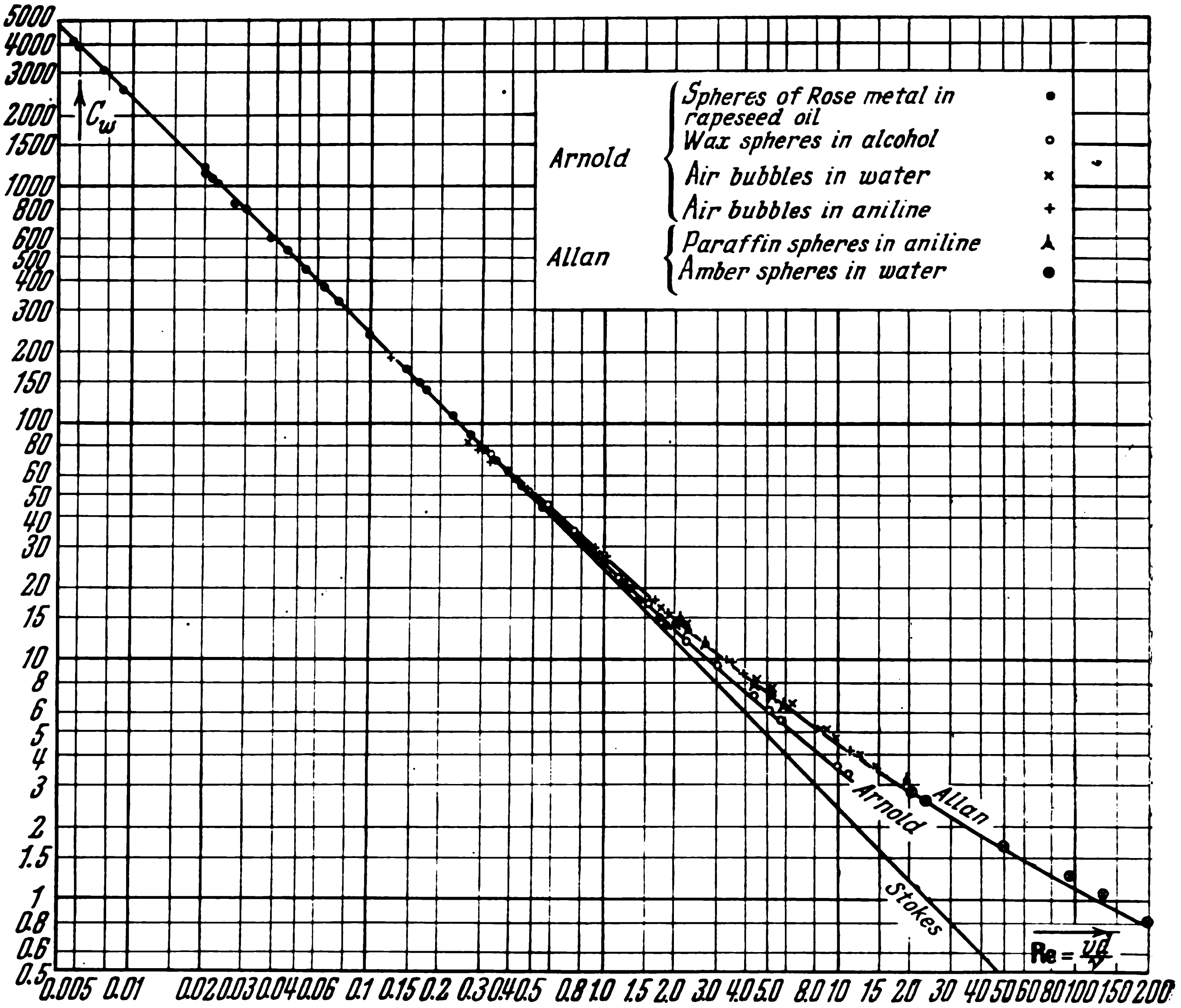

Fig. 9. Drag coefficient of a sphere at low Reynolds numbers.

the inertia property is secondary, giving a high value for μ and a low value for ρ. Formula (4.3) holds in the limit $\rho = 0$; assuming that $\rho \neq 0$, we obtain from this formula

$$W = \rho d^2 v^2 \frac{f_2(\alpha)}{v\, d\rho/\mu}$$

Hence, it follows that

$$f(\alpha, \mathrm{Re}) = \frac{f_2(\alpha)}{\mathrm{Re}} \tag{4.4}$$

at low values of the Reynolds number. This relation is a consequence of Stokes' law. A comparison of experimental data with Stokes' law for a sphere is shown in Fig. 9.

Using dimensional analysis, we have shown that Stokes' law (4.3) is correct for bodies of any shape if the inertia terms in the Navier-Stokes equations are neglected.

The function $f_2(\alpha)$ can be determined experimentally or theoretically by solving the simplified Navier-Stokes equations.

§ 5. Heat Transfer from a Body in a Fluid Flow

In 1915, Rayleigh applied dimensional analysis to the Boussinesq problem of heat transfer from a body to a fluid flowing around the body [2]. Subsequently, the Rayleigh reasoning was the subject of remarks by a number of authors [3-5], but the questions raised in these remarks remained to be cleared up.

We consider in detail all the facts which emerge when dimensional analysis is applied to this problem.

The problem can be stated as follows: a steady process of heat transfer takes place between a body of a given fixed shape and an infinite fluid surrounding the body. The body is fixed, the fluid flows around the body and has the constant velocity v far upstream of the body.

Let H be the quantity of heat emitted by the body per unit time. Assuming the fluid to be ideal and incompressible, Rayleigh reasoned as follows. The quantity H is determined by the values of the following parameters: the characteristic dimension l of the body, the velocity v of the fluid far from the body, the temperature difference T between the temperature of the body and that of the fluid far from the body (it is assumed that the body temperature is kept constant), the specific heat c per unit volume of fluid, and the thermal conductivity λ of the fluid. Therefore, we can write

$$H = f(l, v, T, c, \lambda)$$

Rayleigh chooses the length L, time T, temperature C°, quantity of heat Q, and mass M as fundamental units. Consequently, the dimensions of the parameters will be

$$[l] = \mathrm{L}, \quad [v] = \frac{\mathrm{L}}{\mathrm{T}}, \quad [T] = \mathrm{C}^\circ, \quad [c] = \frac{\mathrm{Q}}{\mathrm{L}^3\mathrm{C}^\circ}, \quad [\lambda] = \frac{\mathrm{Q}}{\mathrm{LC}^\circ\mathrm{T}}$$

We note that all these dimensions are independent of the mass.

Only one independent dimensionless combination

$$\frac{lvc}{\lambda}$$

can be formed from the five characteristic dimensional parameters. The dimensions of H will be

$$\frac{\mathrm{Q}}{\mathrm{T}}$$

It is easy to see that the combination $H/(\lambda l T)$ is a dimensionless quantity, consequently,

$$H = \lambda l T f\left(\frac{lvc}{\lambda}\right) \tag{5.1}$$

This formula was obtained by Rayleigh. It follows that the rate of heat transfer is proportional to the temperature difference T and has identical values for different values of v and c provided that the product vc is constant.

Riabouchinsky made the following remark: since the quantity of heat and temperature have the dimensions of energy (temperature is defined in the kinetic theory of gases as the average kinetic energy of molecules in random motion), then only the units for length, time, and mass can be taken as the fundamental units. The dimensions of the characteristic parameters will then be

$$[l] = \mathrm{L}, \quad [v] = \frac{\mathrm{L}}{\mathrm{T}}, \quad [T] = \frac{\mathrm{ML}^2}{\mathrm{T}^2}, \quad [c] = \frac{1}{\mathrm{L}^3}, \quad [\lambda] = \frac{1}{\mathrm{LT}}$$

Now, the two independent dimensionless combinations

$$\frac{lvc}{\lambda} \quad \text{and} \quad cl^3$$

can be formed from the characteristic parameters. Therefore, in this case, dimensional analysis leads to the formula

$$H = \lambda l T f\left(\frac{lvc}{\lambda}, \quad cl^3\right) \tag{5.2}$$

which clearly yields less information than formula (5.1).

In his answer to Riabouchinsky, Rayleigh wrote [4]:

"Question raised by Dr. Riabouchinsky belongs rather to the logic than to the use of the principle of similitude with which I was mainly concerned. It would be well worthy of discussion. The conclusion I gave follows on the basis of the usual Fourier equations for conduction of heat, in which heat and temperature are regarded as *sui generis*. It would indeed be a paradox if further knowledge of the nature of heat afforded by the molecular theory put us in a worse position than before in dealing with a particular problem. The solution would seem to be that the Fourier equations embody something as to the nature of heat and temperature which is ignored in the alternative argument of Dr. Riabouchinsky." Bridgman [5] correctly noted that Rayleigh's answer is hardly satisfactory, but did not clear up the question himself.

The misunderstanding is explained as follows: there are three different units for energy in the system used by Rayleigh to derive (5.1): the erg $= \mathrm{ML}^2/\mathrm{T}^2$, the degree C°, and the calorie Q. The definition of heat and temperature as mechanical energy is given in the kinetic theory of gases. The conversion of the quantity of heat

and temperature into mechanical units is related to the values of the mechanical equivalent of heat $J = 427$ kgf-m/kcal ($[J] = ML^2/T^2Q$) and the Boltzmann constant $k = 1.38 \times 10^{-16}$ erg/K ($[k] = ML^2/T^2C^\circ$). These must be regarded as physical constants for the independent units for mechanical energy, the quantity of heat, and temperature.

Since the fluid is ideal and incompressible, it follows that the velocity field is determined by kinematic conditions and the phenomenon is not accompanied by a conversion of thermal energy into mechanical energy. The mechanical processes occur independently of the thermal processes. Hence, it follows that the value of the density of the fluid does not affect all the thermal quantities and the value of the mechanical equivalent of heat is generally not essential because of the absence of a conversion of thermal energy into mechanical energy. Furthermore, if it is assumed that the density ρ and the quantity J do not affect the process of heat transfer, then it can be shown by dimensional analysis that the value of the Boltzmann constant k is not at all essential since the dimensions of k include the unit of mass which does not appear in the dimensions of both H and the characteristic parameters. The insignificance of the quantities ρ, J, and k under the above assumptions can also be easily established from the mathematical formulation of the heat transfer problem. These facts justify the omission of ρ, J, and k from the group of characteristic parameters considered by Rayleigh. (If we analyse the same problem in the case of a viscous compressible fluid, then the quantities ρ, J, and k become significant, and these parameters or their equivalents must be included in the table of characteristic parameters.) However, if we maintain the assumption that ρ is insignificant (Riabouchinsky retained this assumption in his reasoning) but make no such assumption about J and k, then the quantities J and k must be added to Rayleigh's table of characteristic parameters, so that the complete system is

$$l,\ v,\ T,\ c,\ \lambda,\ J,\ k$$

From these seven dimensional quantities we can only form the two independent dimensionless combinations

$$\frac{lvc}{\lambda} \quad \text{and} \quad \frac{Jcl^3}{k}$$

In this case, formula (5.1) is replaced by

$$H = \lambda l T f\left(\frac{lvc}{\lambda},\ \frac{Jcl^3}{k}\right) \tag{5.3}$$

Formula (5.3) reduces to (5.1) if we take account of the fact that the mechanical equivalent of heat J is insignificant and, therefore,

that the same is true of the parameter

$$\frac{Jcl^3}{k}$$

Now, if thermal quantities are defined in mechanical terms, following Riabouchinsky, then J and k will be dimensionless universal constants, and formula (5.3) transforms into (5.2). This conclusion is weaker because the methods of analysis here do not take into account the additional reasoning on the mechanism of the phenomena.

§ 6. Dynamic Similarity and Modelling of Phenomena

Dimensions and similarity theory is of considerable importance in modelling various phenomena. *This modelling (simulation) is used to replace the study of the natural phenomenon of interest by the study of an analogous phenomenon in a model of smaller or greater scale, usually under special laboratory conditions.* The basic idea of modelling is that the information required about the nature of effects and various quantities related to a natural phenomenon can be derived from the results of experiments with models.

Modelling is normally based on an analysis of physically similar phenomena. We replace the study of the natural phenomenon of interest to us by the study of a physically similar phenomenon which is more convenient and easier to reproduce. Mechanical or, generally, physical similarity can be considered as a generalization of geometrical similarity. Two geometric figures are similar if the ratios of all the corresponding lengths are identical. If the similarity ratio, the scale, is known, then simple multiplication of the dimensions of one geometric figure by the scale factor yields the dimensions of the other, its similar, geometric figure.

There are various ways of defining mechanical or physical similarity (scaling). Below, we shall give a definition of physical similarity in a form required in practical application and convenient for direct use.

Two phenomena are similar if the characteristics of one can be obtained from the assigned characteristics of the other by a simple conversion, which is analogous to the conversion from one system of units to another. The scaling factor must be known in order to accomplish the conversion.

The numerical characteristics of two different but similar phenomena can be considered as the numerical characteristics of the same phenomenon expressed in two different systems of units. All the dimensionless characteristics (dimensionless combinations of dimensional quantities) of a set of similar phenomena have the same numerical values. It is not difficult to see that the converse is also

correct, i.e. if all the dimensionless characteristics of two motions are identical, then the motions are similar.

A set of mechanically similar motions constitutes a mode of motion.

The similarity of two phenomena can sometimes be understood in a broader sense by assuming that the above definition refers only to a certain special system of parameters. These parameters define the phenomena completely and enable us to find any other characteristics, although they could not be obtained by simple scaling when transforming from one to the other "similar" phenomenon. For example, any two ellipses can be considered similar in this sense when the Cartesian coordinates directed along the major and minor axes of the ellipses are used. The Cartesian coordinates of points of any ellipse can be obtained in terms of the coordinates of points of some particular ellipse (affine similarity) if we use the above conversion.

In order to maintain similarity in modelling, it is necessary to comply with certain conditions. However, quite often in practice, these conditions, which guarantee the similarity of phenomenon as a whole, are not fulfilled, and we must then consider the question of the magnitude of the errors (scale effect) which arise in applying the model results to actual conditions.

After the system of parameters defining the particular class of phenomena has been established, it is not difficult to establish the similarity conditions of the two phenomena.

In fact, let a phenomenon be defined by n parameters some of which can be dimensionless and others dimensional physical constants. Furthermore, let us assume that the dimensions of the variable parameters and of the physical constants are expressed by means of k fundamental units ($k \leqslant n$). In the general case, it is evident that not more than $n - k$ independent dimensionless combinations can be formed from the n quantities. All the dimensionless characteristics of the phenomenon can be considered functions of these $n - k$ independent dimensionless combinations. Therefore, a certain basic system, which defines all the remaining quantities, can be selected from all the dimensionless quantities formed by the characteristics of the phenomenon.

The particular class of phenomena defined by the corresponding formulation of the problem contains phenomena which are not generally similar. The phenomena in a subclass are similar if the following condition is satisfied:

The necessary and sufficient condition for two phenomena to be similar is that the numerical values of the dimensionless combinations forming the basic system are constant. This condition is called similarity criteria.

When the similarity conditions are fulfilled, to calculate the characteristics of the full-scale phenomenon from model data we

need to know the scale factors for the corresponding quantities.

If, among the n characteristic parameters, the k quantities have independent dimensions, then the scale factors for all the k quantities must be arbitrary, and they should be set or determined by the conditions of a problem or derived from experiments. The scale factors for all the remaining dimensional quantities are easily found from formulas relating the dimensions of these quantities to those of the k quantities.

All the dimensionless quantities in the problem of steady uniform motion of a body in an incompressible viscous fluid are defined by the two parameters: the angle of attack α and the Reynolds number Re. The conditions of physical similarity—the similarity criteria—are represented by the relations

$$\alpha = \text{const} \quad \text{and} \quad \mathsf{Re} = \frac{v\,d\rho}{\mu} = \text{const}$$

When modelling this phenomenon, the experimental results for the model can be converted to actual conditions only for identical values of α and Re. The first condition is always easy to realize in practice.

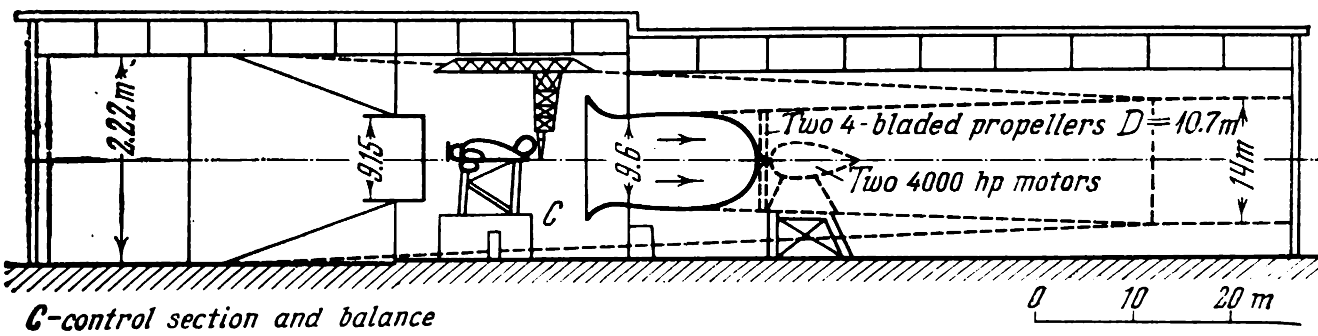

Fig. 10. Longitudinal section of the NASA wind-tunnel for testing under actual conditions. The width of the wind-tunnel is 18.3 m. The return ducts are not shown.

It is more difficult to satisfy the second condition (Re = const), especially in those cases when a streamline body is of a large size as, for example, the wing of an airplane. If the model is smaller than in reality, then either the velocity of the stream must be increased, which is usually restricted in practice, or the density and viscosity of the fluid must be altered substantially in order to maintain the magnitude of the Reynolds number.

In practice, these circumstances introduce great difficulties in the study of aerodynamic drag. The necessity of constant Reynolds number led to the construction of gigantic aerodynamic wind-tunnels in which airplanes could be investigated under actual conditions (Figs. 10 and 11) as well as of closed-type tunnels in which compressed, i.e. more dense, air circulates at high speed (Fig. 12).

Special theoretical and experimental investigations show that on many streamline bodies the Reynolds number noticeably affects

only the dimensionless drag coefficient, but only slightly affects the dimensionless lift coefficient and certain other quantities of practical importance. Therefore, the difference in the values of the Reynolds number for the model and the actual phenomenon is not very important in certain cases.

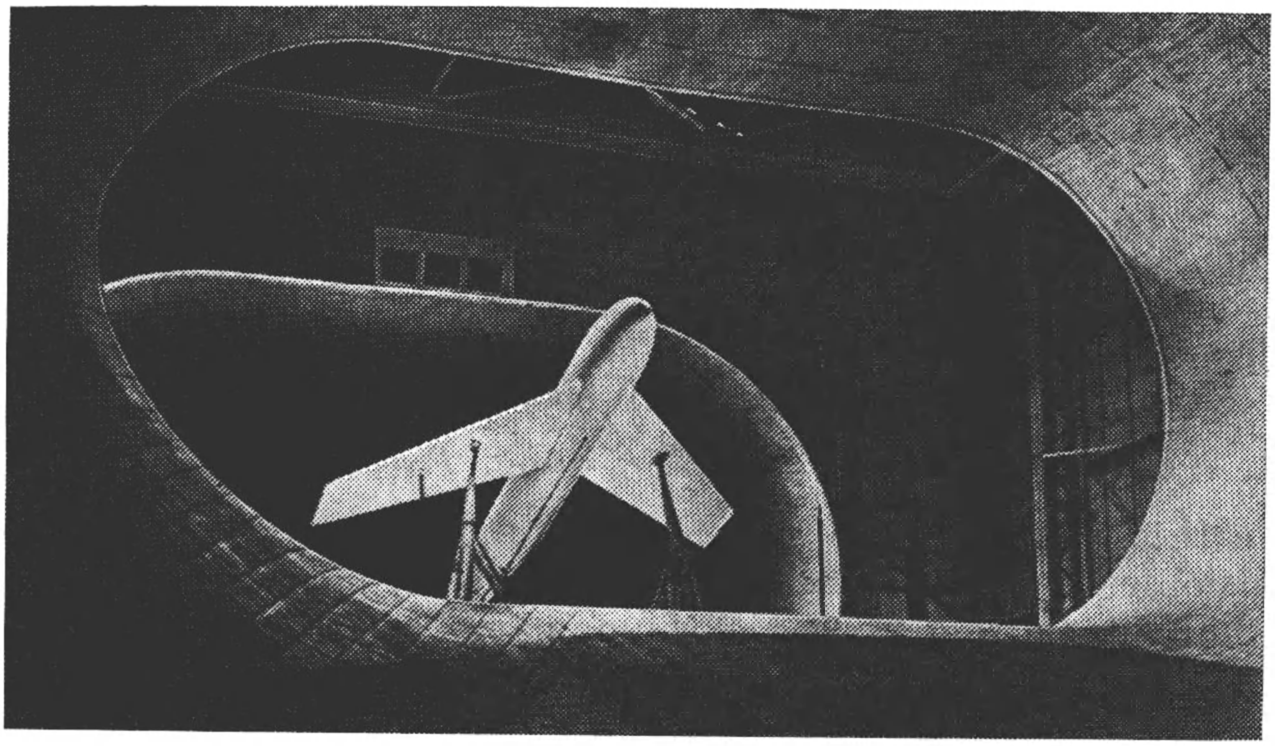

Fig. 11. Photograph of an actual wind-tunnel.

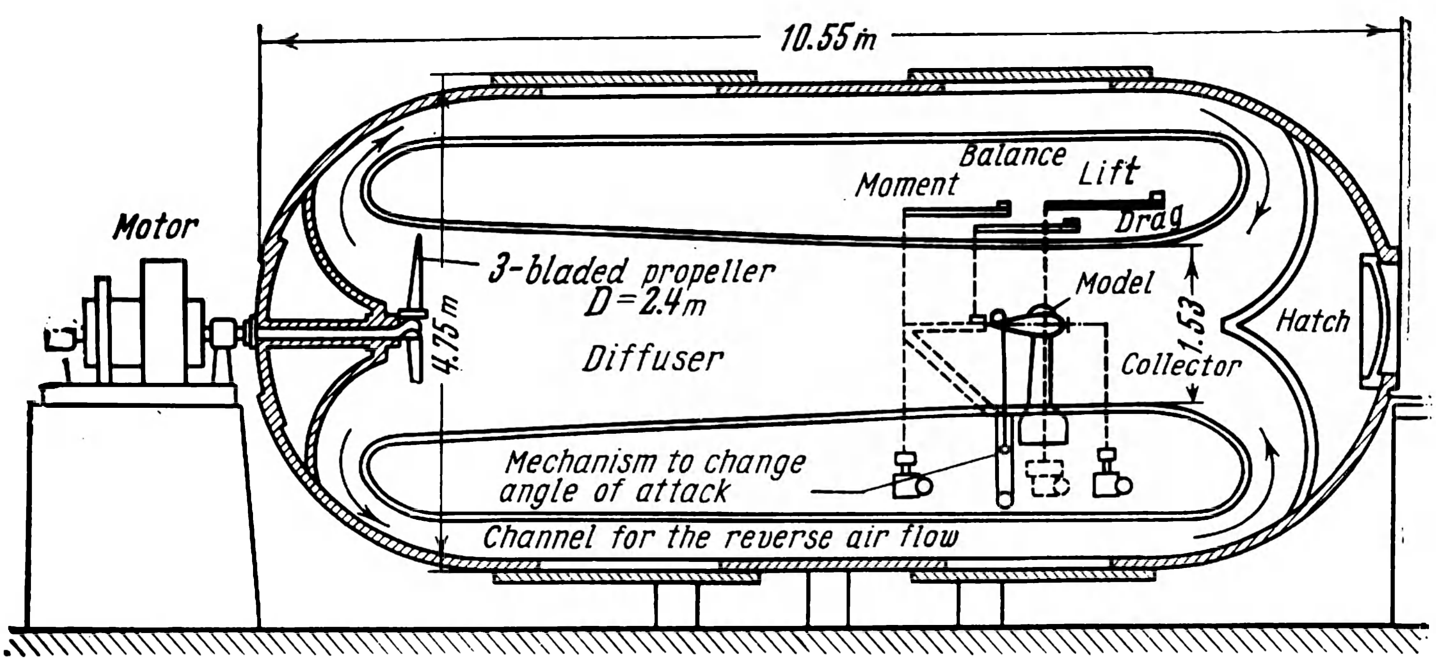

Fig. 12. Cross section of the NASA variable density wind-tunnel. The pressure within the tunnel can reach 21 atm and the velocity of the air flow is 23 m/s.

We mentioned the similarity conditions for wing motion without taking into account the property of compressibility of air; indeed it is not essential for velocities which are low in comparison with the speed of sound. Later, we shall consider the similarity conditions taking compressibility into account.

As another example, let us analyse the problem of modelling the equilibrium of elastic structures.

Consider a structure made of a homogeneous material, for example, a bridge girder. The elastic properties of an isotropic material are determined by the two constants, Young's modulus E kgf/m^2 and the dimensionless Poisson's ratio σ. We consider geometrically similar structures and form a table of characteristic parameters.

In order to define all the model dimensions, it is sufficient to assign a certain characteristic dimension B. If the weight of the structure is essential in the equilibrium state, then the specific gravity $\gamma = \rho g$ kgf/m^3 must appear as a characteristic parameter. External loads distributed in a certain way over the components of the structure act upon it in addition to the weight of its parts. Let the magnitude of these loads be determined by the force P kgf. Then the system of characteristic parameters will be

$$E, \ \sigma, \ B, \ \gamma = \rho g, \ P$$

In this case, we have $n = 5$ and $k = 2$; therefore, the three dimensionless parameters will form the basis for mechanical similarity of the states of elastic equilibrium, namely,

$$\sigma, \quad \frac{E}{\rho g B}, \quad \frac{P}{EB^2}$$

The similarity criteria require that these parameters be equal in the model and in the actual structure. All the deformations wil be similar when these conditions are satisfied. If the model is n times smaller than the actual structure, then the deformations in the model will be n times smaller than in reality.

If the model and the actual structure are produced from the same material, then the values ρ, σ, and E are identical in the model and in reality and, consequently, the following condition must be satisfied for mechanical similarity:

$$gB = \text{const}$$

Under ordinary conditions, $g = \text{const}$; therefore, B must be constant in order to conserve mechanical similarity, i.e. the model must coincide with the actual structure; in other words, modelling is impossible for constant g. (The increase in the specific gravity ρg required in practice when the model dimensions are diminished can sometimes be accomplished by applying an additional load to the model elements.)

A change in g can be realized artificially if the model is forced to rotate at a constant angular velocity by being placed in a so-called centrifuge (Fig. 13). The centrifugal forces of inertia of the model elements can be considered to be parallel for small enough model dimensions and large enough radius of rotation. Performing the

rotation about the vertical axis, we find that constant mass forces, similar to the force of gravity but with another acceleration, will act on the model which is in a state of relative equilibrium (with respect to the centrifuge). Any large value can be obtained for the acceleration by choosing the right angular rotational velocity.

The idea of using centrifuges for modelling various processes was proposed by P. B. Bucky [6] and independently by N. N. Davidenkov [7] and G. I. Pokrovsky [8]. Prior to this in 1929, Davidenkov

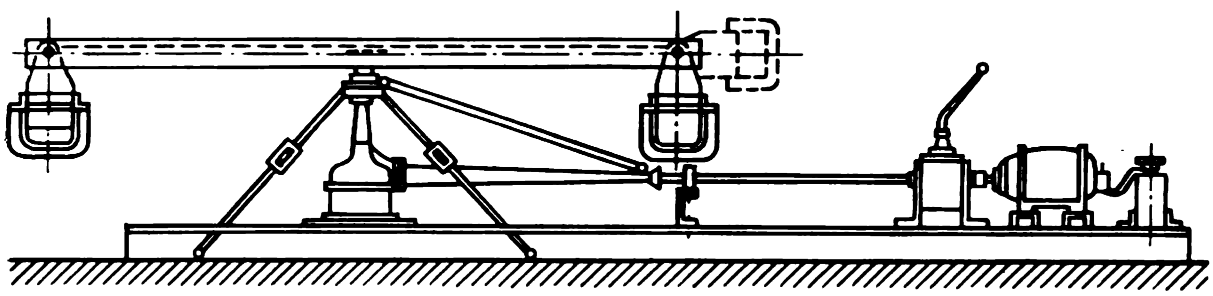

Fig. 13. Schematic diagram of a centrifuge to test models.

[9] proposed to use a box falling on a rigid spring, for this purpose, but this method proved inconvenient and was abandoned.

At the present time, centrifuges have been constructed to investigate various processes by means of models. (The condition $E/(\rho g B) = $ const must be satisfied when modelling processes in which the parameters ρ, g, B, and E, as well as the other essential parameters, are encountered. Consequently, modelling is possible in all such cases by using a centrifuge.)

We consider the stress τ kgf/m^2 which develops in an elastic structure under the action of its weight and a given load distribution. We can assume τ to be the maximum value of some stress component or, in general, to be a certain stress component acting on a specific element of the structure.

The combination τ/E is dimensionless; consequently, we can write

$$\frac{\tau}{E} = f\left(\sigma, \frac{E}{\rho g B}, \frac{P}{EB^2}\right)$$

If the model and the actual structure are made out of the same material, then $E = $ const; consequently, the stress in the corresponding points will be identical in mechanically similar states.

If we assume that the stressed states are mechanically similar and that failure is determined by the values of the maximum stresses, then it is evident that failure occurs both in the model and in reality for similar states. If the magnitudes of the external loads are large but the specific weight of the structure is small enough and can be neglected, then the parameters $\gamma = \rho g$ and, therefore, $E/(\rho g B)$ are

not essential. In this case, the preceding relation becomes

$$\frac{\tau}{E} = f\left(\sigma, \ \frac{P}{EB^2}\right)$$

and the similarity conditions will reduce to the two conditions

$$\sigma = \text{const} \quad \text{and} \quad \frac{P}{EB^2} = \text{const}$$

Hence, it follows that the external loads must be proportional to the square of the linear dimension when modelling with the material properties conserved.

We denote the change in the length under the deformation (strain) of a certain element of an elastic system by l. For a structure of the above type, we have

$$\frac{l}{B} = \varphi\left(\sigma, \ \frac{\rho g B}{E}, \ \frac{P}{EB^2}\right)$$

In a number of cases, it is seen at once from physical considerations that the quantity l/B will diminish with the specific gravity of the basic structure, i.e. with the parameter $\rho gB/E$.

Now, consider two geometrically similar structures of different dimensions but made from the same material (E and σ are identical). Let us assume that the magnitudes of the external loads are proportional to the square of the dimension, i.e.

$$\frac{P}{EB^2} = \text{const}$$

Evidently, the parameter $\rho gB/E$ decreases in this case with the dimensions of the structure; therefore, mechanical similarity will be violated. The strain will be less in structures of smaller dimensions; consequently, a structure of small dimensions will have greater strength. However, this conclusion is valid only when the specific gravity of the material $\gamma = \rho g$ plays an essential part. If the specific weight (γ) is not essential and $P/(EB^2) = \text{const}$, then the strain has the same values for bodies of different scales.

We consider the case when γ is not essential and it is known that the ratio l/B for a given structure decreases with the external loads P. If the external loads are proportional to the cube of the linear dimension, then for structures of small dimensions the ratio l/B will be less than for structures of large dimensions. Therefore, the decrease in the dimensions also increases the strength in this case.

An interesting example in which the scale of the model is defined uniquely is found in hydrostatic models of dirigibles and balloons [10, 11].

In practice, it is very important to know the shape and deformation of elements of the fabric of the balloon after it has been filled

with gas. The geometric shape of the balloon determines its hydrostatic and aerodynamic properties; information on the deformation of the material is necessary to guarantee the strength of the balloon.

We shall outline the parameters which determine the static state of the balloon.

It can be assumed that the air and the gas have constant specific gravities over the height range between the upper and lower surfaces of the balloon. The difference between the air pressure and the gas pressure within the balloon acting on its hull (only the differences in these pressures are essential in this problem) is determined by the magnitude of γ^* equal to the difference in the specific gravities of the air and of the gas, $\gamma^* = \gamma_{\text{air}} - \gamma_{\text{gas}}$.

Experiments investigating the relation between stress and strain in materials show that the strain is identical for a given material when the stress is the same. The stress in the fabric is defined as a force calculated per unit cross-sectional area.

We denote by τ kgf/m a certain stress characterizing the properties of a material, and by l a characteristic linear dimension. Furthermore, let us introduce the weight q kgf/m^2 per unit area of the material, and also the given concentrated external forces Q kgf applied to various elements of the hull (the direction of these forces must be identical in different cases).

We obtain the following system of characteristic parameters for the geometrical similarity of hulls manufactured from materials with the same stress-strain relations:

$$\gamma^*,\ \tau,\ l,\ q,\ Q$$

The similarity conditions are

$$\frac{\gamma_1^* l_1^2}{\tau_1} = \frac{\gamma_2^* l_2^2}{\tau_2} \tag{a}$$

$$\frac{\gamma_1^* l_1}{q_1} = \frac{\gamma_2^* l_2}{q_2} \tag{b}$$

$$\frac{\gamma_1^* l_1^3}{Q_1} = \frac{\gamma_2^* l_2^3}{Q_2} \tag{c}$$

Condition (c) can easily be satisfied by an appropriate choice of the external forces Q, using loads and blocks. If the model and the actual object ($l_2 \neq l_1$) are made from an identical material, then $q_1 = q_2$ and $\tau_1 = \tau_2$, and conditions (a) and (b) contradict each other. Consequently, we confine ourselves to the case when weight of the hull is ignored so that condition (b) drops out. In order to satisfy condition (a) it is necessary to put

$$\frac{l_1}{l_2} = \sqrt{\frac{\gamma_2^*}{\gamma_1^*}}$$

hence, we must increase the specific lift ($\gamma_2^* > \gamma_1^*$) as the scale is decreased ($l_2 < l_1$). This effect can be accomplished by using a heavy fluid and by suspending the model "upside down". The hydrostatic models of dirigibles and balloons use this idea. Water, mercury, glycerin, etc. can be taken as the filling for the model. The usual modelling arrangement is shown in Fig. 14.

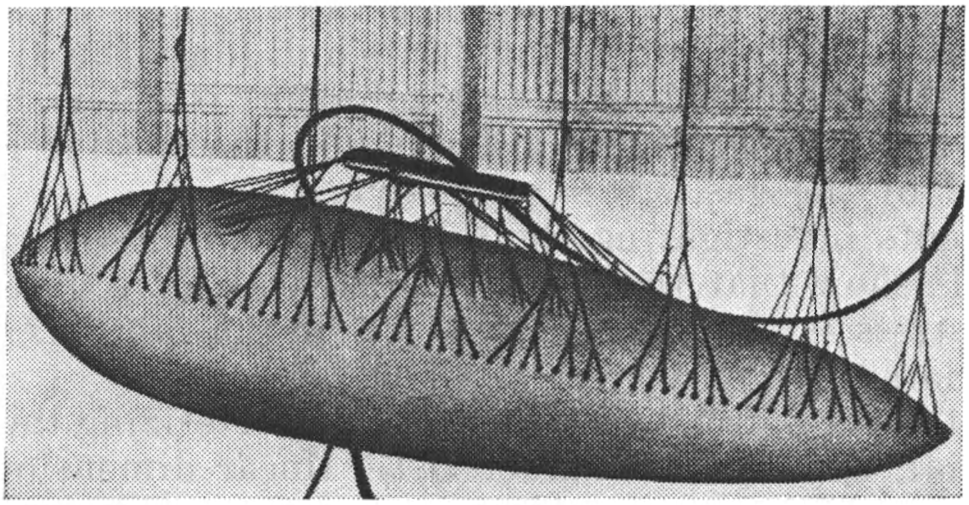

Fig. 14. Testing of an airship model.

If the flight altitude, temperature, pressure, and the gas used are known for the actual case, then by choosing the fluid for the model, we obtain the ratio of the specific gravities, which defines the scale of the model.

For example, we have under normal conditions:

for hydrogen . . . . $\gamma_1^* = 1.1$ kgf/m³
for water . . . . . $\gamma_2^* = 1000$ kgf/m³
for mercury . . . . $\gamma_2^* = 13{,}600$ kgf/m³

Hence, we find the following value for the model scale when using water

$$\frac{l_1}{l_2} = n = \sqrt{\frac{1000}{1.1}} = 30.1$$

when using mercury, we obtain

$$n = \sqrt{\frac{13{,}600}{1.1}} = 111$$

Modelling with mercury leads to smaller models, which generally is inconvenient.

The weight of the actual hull and the weight of the model hull act in opposite ways; consequently, the effect of the weight will violate similarity. The effect of the weight of the hull can be taken into account with the aid of special instrumentation by applying

external forces acting vertically upward and distributed over the hull in conformity with similarity condition (b), taking account of the specific weight of the hull directed downward.

Sometimes, modelling can be carried out by using *a fortiori* non-similar phenomena when certain similarity criteria Π_1, Π_2, . . . have different values in the model and in reality, provided that the relations of the desired dimensionless quantities and these dimensionless parameters are known beforehand from additional considerations. In such cases, it is only necessary to keep those similarity criteria constant whose relation is not known.

The above type of modelling can be used when the relations between the required quantities and the parameters Π_1, Π_2, . . . are assumed in advance. Their validity can be confirmed or refuted by means of the model investigations. As has been mentioned above, an example of this is the modelling at different values of the Reynolds number when its effect on the unknown in question is insignificant. However, this same method can be used in those cases when the Reynolds number is essential but its effect is known beforehand.

An investigation with models is often the only possible method of experimental study and of solving important practical problems. Such is the case when studying natural phenomena which proceed through decades, centuries or millennia; a similar phenomenon can be produced under model experimental conditions for only several hours or days. This is the case in modelling the phenomenon of oil seepage through rock when oil is pumped out through a well. Conversely, instead of investigating a phenomenon which occurs very rapidly in nature, we can study a similar phenomenon which occurs much more slowly in a model.

Modelling is a recognized scientific technique which has general value, in principle and in practice, but it must be regarded only as the initial approach to the main problem. This problem is the actual determination of the laws of nature, the search for general properties and characteristics of various classes of phenomena, the development of experimental and theoretical methods of investigation and of solution of various problems, and, finally, obtaining systematic techniques, methods, rules, and recommendations to solve specific practical problems.

§ 7. Steady Motion of a Solid Body in a Compressible Fluid

We now consider the general problem of the steady forward motion of a solid body at a constant velocity in an infinite fluid. We shall take into account the inertia, viscosity, compressibility, and heat conduction of the fluid. For simplicity, we shall not consider the weight of the fluid (actually, the weight of fluid may be essential

in a number of cases since it can cause intense convective motion arising from the nonuniform heating of the fluid) and heat transfer due to radiation.

Let us formulate the problem mathematically in order to determine the system of characteristic parameters. First, we write down the equations of motion of a compressible viscous fluid, which we shall consider to be an ideal gas. We shall have

(1) the Navier-Stokes equation

$$\rho \frac{d\bar{\mathfrak{v}}}{dt} = -\operatorname{grad} p - \frac{2}{3} \operatorname{grad} \mu \operatorname{div} \bar{\mathfrak{v}} + 2 \operatorname{div} \mu \operatorname{def} \bar{\mathfrak{v}} \tag{7.1}$$

where def $\bar{\mathfrak{v}}$ is the tensor of the strain rate;

(2) the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \bar{\mathfrak{v}} = 0 \tag{7.2}$$

(3) the equation of state of a gas (the Clapeyron equation)

$$p = \rho R T \tag{7.3}$$

where R is the gas constant;

(4) the equation of heat transfer

$$J c_V \rho \frac{dT}{dt} + p \operatorname{div} \bar{\mathfrak{v}} = \operatorname{div} (J \lambda \operatorname{grad} T) - \frac{2}{3} \mu (\operatorname{div} \bar{\mathfrak{v}})^2$$
$$+ \mu \left\{ 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \right.$$
$$\left. + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right\} \tag{7.4}$$

The left-hand side of this equation gives the change in the internal energy due to temperature, and the work of the pressure forces. The right-hand side gives the change in the energy due to the influx of heat contributed by heat conduction and by the work of the internal frictional forces.

Further, we shall assume that the thermal conductivity and the coefficient of viscosity change with temperature according to the Sutherland formula [12]

$$\frac{\lambda}{\lambda_0} = \frac{\mu}{\mu_0} = \frac{1 + \frac{C}{273.1}}{1 + \frac{C}{T}} \sqrt{\frac{T}{273.1}} \tag{7.5}$$

where λ_0 and μ_0 are the thermal conductivity and the coefficient of viscosity at the temperature $T = 273.1$ °C corresponding to zero in the Centigrade scale, and C is Sutherland's constant, which has the dimensions of temperature. The constant C has different values for different gases. For air $C \approx 113$ °C.

Besides these equations, we have the conditions at infinity and the boundary conditions: at infinity in front of a body and in directions perpendicular to its velocity, the fluid is at rest, $|\bar{\mathfrak{v}}|_\infty = 0$, but the density and the absolute temperature have the assigned values ρ_0 and T_0.

The shape of the body surface is fixed; consequently, all body dimensions are determined by the value of a certain characteristic dimension l.

The kinematic conditions of flow are completely defined by the given value of the forward velocity of the body v and by the two angles α and β that the velocity vector forms with the axes of the coordinate system fixed to the body.

Finally, the boundary conditions for the temperature on the body surface must be given. (In many technical problems and in aerodynamics in particular, we encounter problems of fluid flow around a strongly heated or cooled body; for example, the flow around engine radiators, the investigation of the flow around cooled cases of aircraft, etc.) Let us assume that the body temperature is constant on the surface and equal to T_1.

Equations (7.1)-(7.5), the conditions at infinity, and the boundary conditions show that the system of characteristic parameters is

$$l,\ \alpha,\ \beta,\ v,\ T_0,\ T_1,\ \rho_0,\ \mu_0,\ R,\ c_V,\ J,\ \lambda_0,\ C$$

The dimensions of the thirteen quantities listed are expressed in terms of the five basic units L, T, M, Q, and C°.

The number of parameters can be cut down by one if account is taken of the fact that the constants J, c_V, and λ enter into (7.4) only as the products Jc_V and $J\lambda$. If the three parameters J, c_V, and λ were to be replaced by the two parameters Jc_V and $J\lambda$, then the dimensions of all the quantities would be expressed in terms of only the four fundamental units L, T, M, and C°; consequently, such a reduction in the number of parameters is not essential. The type of motion and the class of similar motions are determined by the values of the eight independent dimensionless parameters:

$$\frac{R}{Jc_V} = \gamma - 1, \quad \frac{\mu c_V}{\lambda_0} = P, \quad \alpha,\ \beta, \quad \frac{\rho_0 l v}{\mu_0} = \mathsf{Re},$$

$$\frac{v}{\sqrt{\gamma R T_0}} = \mathsf{M}, \quad \frac{T_1}{T_0}, \quad \frac{C}{T_0}$$

where M is the Mach number.

It is shown in the kinetic theory of gases that the constants γ and P depend only on the structure of the gas molecules. The following theoretical formulas are known for an ideal gas:

$$\frac{c_p}{c_V} = \gamma = 1 + \frac{2}{m}, \quad \frac{\lambda}{\mu c_V} = \frac{9\gamma - 5}{4} \quad \text{(A. Euchen)}$$

where m is the number of degrees of freedom of the gas molecule. For a monatomic gas, $m = 3$; for a diatomic gas, $m = 5$. The experimental value for many gases (helium, hydrogen, and some others) is close to the theoretical prediction, but the effect of temperature and pressure is noticeable when they change over a wide range [13]. In practice, we can assume that for air at normal temperature and pressure

$$\gamma = 1.4, \qquad \frac{\lambda}{\mu c_V} = P^{-1} = 1.9, \qquad \frac{\mu c_p}{\lambda} = 0.737$$

The condition that motions of geometrically similar bodies be mechanically similar is that the above eight parameters be constant. Let us denote the total drag by W, and the lift by A. For gases with identical constant values of the numbers γ and P, we have

$$\frac{2W}{\rho l^2 v^2} = c_x = f_1\left(\alpha,\ \beta,\ \mathrm{Re},\ \mathrm{M},\ \frac{T_1}{T_0},\ \frac{C}{T_0}\right)$$

$$\frac{2A}{\rho l^2 v^2} = c_y = f_2\left(\alpha,\ \beta,\ \mathrm{Re},\ \mathrm{M},\ \frac{T_1}{T_0},\ \frac{C}{T_0}\right)$$

In addition to the above eight dimensionless parameters, it is necessary to introduce the dimensionless coordinates of the point in space x/l, y/l, and z/l (to be definite, we assume that the coordinate system is fixed to the body) when studying the pressure distribution, temperature field, velocity field, etc. For example, for the pressure and temperature distribution, we have

$$\frac{2p}{\rho_0 v^2} = f_3\left(\frac{x}{l},\ \frac{y}{l},\ \frac{z}{l},\ \alpha,\ \beta,\ \mathrm{Re},\ \mathrm{M},\ \frac{T_1}{T_0},\ \frac{C}{T_0}\right)$$

$$\frac{T}{T_0} = f_4\left(\frac{x}{l},\ \frac{y}{l},\ \frac{z}{l},\ \alpha,\ \beta,\ \mathrm{Re},\ \mathrm{M},\ \frac{T_1}{T_0},\ \frac{C}{T_0}\right)$$

The determination of the form of the functions f_1, f_2, f_3, and f_4 is the basic problem of experimental and theoretical aerodynamics.

The parameter C/T_0 appears to be essential only when heat conduction and viscosity play an appreciable part, and the phenomenon is characterized by noticeable temperature changes. But even when heat conduction and viscosity are important, the effect of temperature on the coefficient of viscosity and thermal conductivity can usually be represented by the following formula (which does not contain the dimensional constant C) as a good approximation in place of formula (7.5)

$$\frac{\lambda}{\lambda_0} = \frac{\mu}{\mu_0} = \sqrt{\frac{T}{273.1}}$$

or by the more general formula

$$\frac{\lambda}{\lambda_0} = \frac{\mu}{\mu_0} = \left(\frac{T}{273.1}\right)^k$$

where k is an arbitrary constant. These arguments show that the parameter C/T_0 does not play an essential part in practice and, therefore, it can be neglected in modelling.

When the velocity of a body is small (small M), the temperatures T_1 and T_0 only differ slightly, so that the ratio T_1/T_0 is close to unity.

In the previous discussions, we assumed that the body temperature T_1 has a given value. If the transfer of heat from a body to a fluid can be neglected, then the boundary conditions at the body surface can be written as

$$\frac{\partial T}{\partial n} = 0$$

This corresponds to the case when the body surface is thermally insulating; under this condition, the parameter T_1/T_0 is eliminated. The same applies when heat conduction is neglected completely and adiabatic processes are considered.

Sometimes the parameter T_1/T_0 is eliminated because the temperature T_1 becomes a characteristic quantity. For example, in some cases, because of the heat transfer from a fluid to a body, the body temperature builds up and reaches a certain value different from the fluid temperature at infinity, independently of the thermal properties of the body. Such a case arises when measuring the temperature of a gas, moving at a very high speed, with a liquid-filled thermometer. The thermometer readings depend on Re and M and on the method of orienting the thermometer relative to the flow; the body (thermometer) temperature will differ from the undisturbed stream temperature far from the body.

With the above-described additional data taken into account in the formulation of the problem, the constancy of the Reynolds and Mach numbers (the Mach number being essential only if compressibility is significant) is obviously of paramount importance for the similarity of perturbed gas motions.

All the above conclusions can easily be extended to the case of propeller motion in a fluid. Besides the forward motion of the propeller, rotation about the axis also enters into the propeller problem. Consequently, still one more parameter, the angular velocity, which can be defined by the number of revolutions n of the propeller per unit time, is added in the steady motion of a propeller with constant forward and angular velocities. The dimensionless parameter $v/(nl)$, which is called the advance ratio of the propeller, is added to the characteristics of the forward motion of a body in the propeller motion case. This parameter is fundamental when studying the aerodynamic or hydrodynamic properties of propellers.

If a fluid is ideal and incompressible, then the advance ratio is the single dimensionless parameter defining the flow due to a propeller moving forward along its axis at a uniform rate (at a constant pitch).

§ 8. Unsteady Motion in a Fluid

The conclusions of dimensional analysis which were obtained for steady motion in the previous paragraph can be generalized to unsteady motion.

In the general case, when studying unsteady motion, we must introduce the time t, which is variable, into the series of characteristic parameters. In analysing mechanically similar motions, we find that t changes with scale and with time during the motion. In this connection, we shall first make some general remarks about kinematically similar unsteady motions.

In such motions all appropriate dimensionless combinations formed from the kinematic quantities are identical for all motions with kinematic similarity. The class of motions can be broadened to include kinematically dissimilar motions if it is assumed that certain dimensionless kinematic parameters characterizing the whole motion can assume different values for different motions.

Each particular motion and the type of kinematically similar motions as a whole are defined by three parameters. Two parameters classify the motion of the system as a whole and one parameter fixes the particular state of the motion.

A length characterizing the geometric dimensions and a characteristic time can be taken as the parameters defining the particular motion. These parameters determine the kinematic scales of the law of motion.

For example, it is natural to select the radius of a circle as the characteristic dimension in motion along the circle, and the amplitude of oscillations of a specified point in oscillatory motion. It is natural to select the period as the characteristic time for periodic processes. The velocity for a certain particular state or mean velocity can be taken instead of the characteristic time.

The value of the time t determines the instantaneous state of a given unsteady motion. (The time origin and the origin for linear coordinates are selected so that the position of the system and the state of motion at $t = 0$ be similar for different motions.)

Let d, v, and t be the characteristic length, the characteristic velocity, and the moment of time under consideration. Similar or, in other words, corresponding states of the motion, for motions similar as a whole, are determined by the value of the dimensionless variable parameter vt/d which can be considered a dimensionless time.

The relation $v_1t_1/d_1 = v_2t_2/d_2$, if satisfied for similar states of motion of two systems, can be considered the time-conversion condition when changing from one system to the other.

As an example, let us consider the system of harmonic oscillations of a point along the arc of a circle of radius d. The law of motion is

$s = a \cos kt$, where s is the arc length. For kinematically similar motions, we must have

$$\frac{a}{d} = \text{const}$$

The particular motion is determined by the amplitude a and by the frequency k. The state of motion is determined by the time t.

Similar states for different motions are characterized by identical values of the dimensionless parameter kt.

If the ratios a_1/d_1 and a_2/d_2 have different values in two motions, then such motions are kinematically dissimilar. The ratio a_1/a_2 defines the length scale. For kinematically similar harmonic oscillations along a circle, we must have $a_1/a_2 = d_1/d_2$. It is impossible to choose a characteristic dimension for a straight line; consequently, any two harmonic motions in a straight line are kinematically similar.

Let us consider the rectilinear motion $s = vt + a \cos kt$ representing uniform motion combined with a harmonic oscillation. In this case, the class of similar motions is characterized by a constant value of the dimensionless parameter $\mathsf{St} = ak/v$, called the Strouhal number. Similar states of motion are characterized by the value of the parameter kt or by the value of the parameter $vt/a = kt/\mathsf{St}$.

The class of motions can be broadened and dissimilar motions can be considered by assuming that the Strouhal number varies for different motions while the basic law of oscillations is retained.

Now let us consider the unsteady motion of a body in a fluid, assuming that the law of motion of the body itself is known. A certain length d and velocity v can be taken as the dimensional parameters defining a particular motion. By comparison with the steady-motion case, the system of characteristic parameters in the unsteady-motion case, with a given law of motion, is supplemented only by the value of the length d characterizing the law of motion and by a variable parameter, the time t. Consequently, the system of dimensionless parameters defining the motion as a whole and each state of the motion is supplemented only by two parameters d/l and vt/l, where it is necessary to impose the condition $d/l = \text{const}$ for the two motions to be similar; the constancy of the parameter vt/l only defines the appropriate values of the time (time scale) for similar states of motion.

If unsteady motions are oscillations with a definite shape and an arbitrary frequency k, then the table of characteristic parameters is supplemented by the parameter k, with the consequent addition of the Strouhal number

$$\frac{kl}{v} = \mathsf{St}$$

as a dimensionless parameter defining the motion.

We consider an unsteady motion of a body in a fluid, namely, the combination of motion with the forward velocity v and an oscillatory motion of fixed shape, but with a variable frequency k. The Strouhal number must be kept constant to ensure the similarity of different motions if k, l, and v are given beforehand in terms of the data of the problem considered. If the frequency k is a specific quantity, then the condition of constant Strouhal number is obtained as a consequence of the similarity conditions formed from the quantities assigned. In many cases, we have to study an unsteady motion of a body in a fluid when the body motion is not known beforehand. As an example, consider the problem of the oscillations of an elastic wing in an advancing stream of fluid (wing flutter).

For simplicity, let us assume that the wing, with a longitudinal plane of symmetry, is clamped rigidly along its centre section.

The elastic properties of wings composed of homogeneous material and of fixed structural design are determined by two physical constants.

We omit the analysis of the practical methods of establishing the elastic properties of airplane wings. A detailed formulation of the problem permits the elastic properties of various structures to be characterized by certain functions and parameters which are essential only from the viewpoint of the problem under consideration. Thus, classes of wings with equivalent elastic properties but different geometric properties and, in general, with different structures can be singled out.

In the general formulation of the problem, we can assume that all the elastic characteristics, within the limits of validity of Hooke's law, of each wing from the different classes of geometrically similar wings with equivalent elastic properties are defined by the value of the characteristic dimension l, Young's modulus E, and the shear modulus G.

The distribution of mass can influence oscillations. The mass distribution law can be expressed in such a form that the mass of each element would be proportional to the total mass and would be defined completely by the value of a certain mass m.

It is well known that the dynamic properties of an ideal rigid body are defined completely by the total mass of the body, by the position of the centre of gravity, and by the inertia tensor at the centre of gravity of the body.

In the practical approximate formulation of the problem, a detailed analysis shows that elastic wings with different mass distributions can, just as in the case of an ideal rigid body, be dynamically equivalent.

In practice, a system of abstract parameters characterizing the distribution of mass is established to determine the arising oscillatory motions (the position of the centres of gravity of various wing

sections, the moments of inertia of sections, etc.). All subsequent conclusions can be extended to the cases of elastic wings with different, but dynamically equivalent, mass distributions.

Let us assume that the fluid is ideal, homogeneous, and incompressible in the phenomenon under consideration. The mechanical properties of the fluid are then determined completely by the density ρ. The effect of gravity on the fluid cannot be taken into account. Furthermore, let us assume that the fluid is infinite. The wing centre section is fixed. The fluid at infinity has the forward velocity v which is constant in time and parallel to the longitudinal plane of symmetry of the wing, the velocity v having a fixed direction relative to the fixed wing section.

Summarizing, we find that the unsteady motion of the wing-fluid system is determined, within the limits of our assumptions, by the parameters

$$l,\ E,\ G,\ m,\ \rho,\ v$$

and the quantities giving the initial disturbance. Moreover, each separate state of motion is determined by the moment of the time t.

Those properties of perturbed unsteady motions, which are unaffected by the nature of the initial perturbations, are evidently independent of the parameters defining these perturbations.

Therefore, these properties are determined, for dynamically similar motions, by the system of dimensionless parameters

$$\frac{\rho v^2}{E},\quad \frac{G}{E},\quad \frac{m}{\rho l^3},\quad \frac{vt}{l}$$

where it is understood that the mass distribution and the wing structure are identical in different cases.

The last property can be considered in a broadened sense and only the equivalence of the elastic and dynamic properties of the wings need be required while retaining the geometrical similarity of the external wing surfaces adjacent to the fluid.

It is evident that the properties characterizing the motion as a whole are independent of the parameter vt/l, whatever the separate states of each motion. Experiment shows that steady flow past an elastic wing can be either stable or unstable.

A decrease in the displacements due to an oscillatory disturbance of a wing in a moving stream occurs in stable motion while an increase in displacements occurs in unstable flow, with possible wing failure as a result.

The stability and instability of the streamlines for sufficiently small perturbations can be considered as properties independent of the initial conditions and of the separate states of motion. Consequently, the properties of stability of motion must be defined by the

system of parameters

$$\frac{\rho v^2}{E}, \quad \frac{G}{E}, \quad \frac{m}{\rho l^3}$$

In the general formulation of the problem, it is impossible to divide the set of all motions into motions with increasing and decreasing amplitudes; cases are possible when maximum deflections are either constant or variable during oscillations but retain small enough values for any t.

If the motions with increasing and decreasing amplitudes can be clearly separated, then the boundary between these modes is defined by the relation

$$F\left(\frac{\rho v^2}{E}, \quad \frac{G}{E}, \quad \frac{m}{\rho l^3}\right)=0$$

which can be written in the form

$$v_{cr}=\sqrt{\frac{E}{\rho}}\, f\left(\frac{G}{E}, \quad \frac{m}{\rho l^3}\right)$$

This formula defines the critical flutter speed. The value of the critical velocity separates the stable and unstable flow modes for variable velocity of incoming flow, with the rest of the parameters being constant.

The wing stiffness is proportional to E and G. An increase in E and G by a factor of n is equivalent to an increase in the stiffness coefficient by a factor of n. It is seen from the formula for v_{cr} that the critical velocity will be increased by a factor of $\sqrt{n}$ when the wing stiffness is increased by a factor of n if the wing mass, shape, and dimensions remain the same.

The Strouhal number kl/v_{cr} corresponding to the critical state is defined by the values of the abstract parameters G/E and $m/(\rho l^3)$ which also define all the abstract quantities independent of the initial data and characterizing the critical motion.

§ 9. Ship Motion

Methods of dimensional analysis and similarity are of great practical value in numerical and experimental solutions of problems of motion on a water surface [14].

We consider the steady rectilinear forward motion of a ship over the surface of a semi-infinite fluid which is at rest at large depth and at a large distance ahead of the ship. The motion of a floating body causes a perturbation of the free surface. The perturbed fluid motion is of wave type and is determined by the effect of gravity.

We shall take into account the important effects of density ρ, the gravity g, and the viscosity μ of water. The compressibility of

water is of no practical significance and is neglected. Neither is capillarity essential to the motion of ordinary ships.

The dimensions and shape of the ship hull have a large effect on the principal mechanical characteristics. Let us first analyse the motion of the ship with a fixed hull shape. All the geometric dimensions are determined by the ship length L. Geometrically similar hulls correspond to different values of L. For ordinary heavy ships, it can be assumed that the total weight completely determines the hull position relative to water. Evidently, the position of the ship relative to water affects the drag, etc. Consequently, we take the weight or the displacement A of the ship as a characteristic parameter. (If different motions occur for similar ship locations relative to the water level, then the weight is proportional to L^3. It is sufficient to take one of the parameters, L or A, in this case.) Instead of displacement A (in tons) the volume displacement D (in cubic metres) can be taken, since $A = \rho g D$, where ρ is the water density. For fresh water, we have

$$\rho g = 1 \ \text{ton/m}^3$$

Let us denote the speed of the ship by v. The system of characteristic parameters will be

$$L, \ D, \ \rho, \ g, \ \mu, \ v$$

(The motion occurs at the interface between water and air; consequently, the air density ρ' and viscosity μ' must enter as characteristic parameters (the air compressibility is not significant at the usual speeds). However, these parameters exert only a slight effect on the phenomenon, and taking them into account does not alter the subsequent conclusions since the dimensionless quantities ρ'/ρ and μ'/μ added can be considered constant for each class of motion.) All the geometrical and mechanical quantities, for example, the wetted area S, the drag W, etc., can be considered functions of these six parameters. (The quantity S differs slightly from the magnitude of the wetted area in the static state, which is determined by only two parameters L and D.) Dynamically similar motions and the state of each motion are defined by the three dimensionless parameters:

$$\psi = \frac{L}{\sqrt[3]{D}} \quad \text{(fineness coefficient)}$$

$$\mathsf{Fr} = \frac{v}{\sqrt{gL}} \quad \text{(the Froude number)}$$

$$\mathsf{Re} = \frac{\rho v L}{\mu} \quad \text{(the Reynolds number)}$$

(The fineness coefficient can be interpreted as the length of a geometrically similar hull with a one-ton displacement.) Hence, we can

write for the drag

$$W = f(\psi, \mathsf{Fr}, \mathsf{Re})\,\rho S v^2 \tag{9.1}$$

The similarity criteria are

$$\frac{L_1}{\sqrt[3]{D_1}} = \frac{L_2}{\sqrt[3]{D_2}}, \qquad \frac{v_1}{\sqrt{gL_1}} = \frac{v_2}{\sqrt{gL_2}}, \qquad \frac{v_1 L_1 \rho_1}{\mu_1} = \frac{v_2 L_2 \rho_2}{\mu_2}$$

It is not difficult to see that if an identical fluid is used both for the model motion and the actual motion, then $\mu_1/\rho_1 = \mu_2/\rho_2$ and modelling of the phenomenon is impossible. Actually, since the Froude number is constant, the velocity must decrease with the linear dimensions of the ship; further, since the Reynolds number is constant, the velocity must increase as the linear dimensions of the ship decrease. Consequently, complete similarity is not maintained when modelling this phenomenon with a change of length scale, and the magnitude of the drag coefficient of the model is not equal to the magnitude of this coefficient in the actual motion.

The determination of ship drag using model experiments is based on the practical possibility of decomposing the drag into two components: one is determined by viscosity and the other is determined by gravity. Indeed, formula (9.1) can be replaced approximately by the following:

$$W = W_1 + W_2 = c_f(\mathsf{Re})\frac{\rho S v^2}{2} + c_w(\psi, \mathsf{Fr})\,\rho g D \tag{9.2}$$

The form of (9.2) can be established by theoretical arguments which we shall not touch upon here.

The drag

$$W_1 = c_f(\mathsf{Re})\,\rho\,\frac{S v^2}{2}$$

is called friction drag. The friction drag for the model motion and for the actual motion is found by calculations based on various semi-empirical formulas. The value of the coefficient of friction drag is determined by the Reynolds number Re. Moreover, this coefficient depends on the roughness of the surface and, to a certain degree, on the shape of the ship's hull contour. The coefficient of friction drag decreases as the Reynolds number increases. In practice, the coefficient c_f is taken equal to the corresponding coefficient on a flat plate. Experimental results on flat plates are given in Fig. 6.

The value of the coefficient c_f for smooth plates is determined according to the Prandtl-Schlichting formula

$$c_f = \frac{0.455}{(\log \mathsf{Re})^{2.58}} \tag{9.3}$$

The value of the coefficient c_f for a rough-surface plate will be considerably higher.

The drag W_2 is called the residual drag. The coefficient $c_w = W_2/(\rho g D)$ gives the residual drag per ton displacement. This coefficient can be determined experimentally, by testing geometrically similar models while complying with the following conditions:

$$\frac{L_1}{\sqrt[3]{D_1}} = \frac{L_2}{\sqrt[3]{D_2}} \quad \text{and} \quad \frac{v_1}{\sqrt{gL_1}} = \frac{v_2}{\sqrt{gL_2}}$$

These conditions are the Froude similarity laws.

The residual drag depends on the hull shape. When studying the effect of the hull shape, it is necessary to broaden the class of motions and to study the motion of a family of hulls the shapes of which depend on several geometric parameters.

In practice, it is very important to single out the geometric parameters which have a significant effect on the residual drag. Experiments show that the basic parameters for all possible geometrically dissimilar ship hull contours of ordinary types which determine the coefficient c_w are the Froude number and the fineness coefficient. The volume-displacement Froude number

$$\mathsf{Fr}_D = \frac{v}{\sqrt{g\sqrt[3]{D}}} = F\sqrt{\psi}$$

can be taken instead of the linear Froude number $\mathsf{Fr} = v/\sqrt{gL}$.

The graph of Doyère [15] giving averaged experimental data on the dependence of the coefficient c_w on ψ and Fr_D for hulls without projecting parts (rudders, cantilevers for the propeller shafts, etc.) is shown in Fig. 15. Using the Doyère graph and the values of the coefficient of friction as a function of the Reynolds number, the ship hull drag can be calculated easily as a function of ship velocity. This computation often gives very good results as a first approximation.

It appears that the following parameters are the most important in making a more exact determination of the residual drag [16]:

$$\chi = \frac{D}{L\varkappa} \quad \text{and} \quad \frac{B}{T}$$

where $\varkappa$ is the midships area, B is the hull width, and T is the draught. The coefficient χ is called the prismatic block coefficient.

Besides the fundamental parameters mentioned, parameters determining the effect of various quantities characterizing the midships shape, the bow, the stern, etc. on the drag can also be taken into account.

When ship motion is studied, the tow drag W' of the ship without propellers and the drag W'' with rotating propellers, which cause a motion-producing thrust, are considered. The drag W in the last case equals the horizontal component of the propeller pull in uniform steady motion. The drag W' is a hull characteristic independent of

the motor properties. Because of the interaction between the hull and the propeller, the inequality $W' < W''$ usually holds.

If the hull, the location of the propeller, ship velocity, ship displacement, and all dimensions are given, then the drag is determined and so are the required number of propeller revolutions n, and the

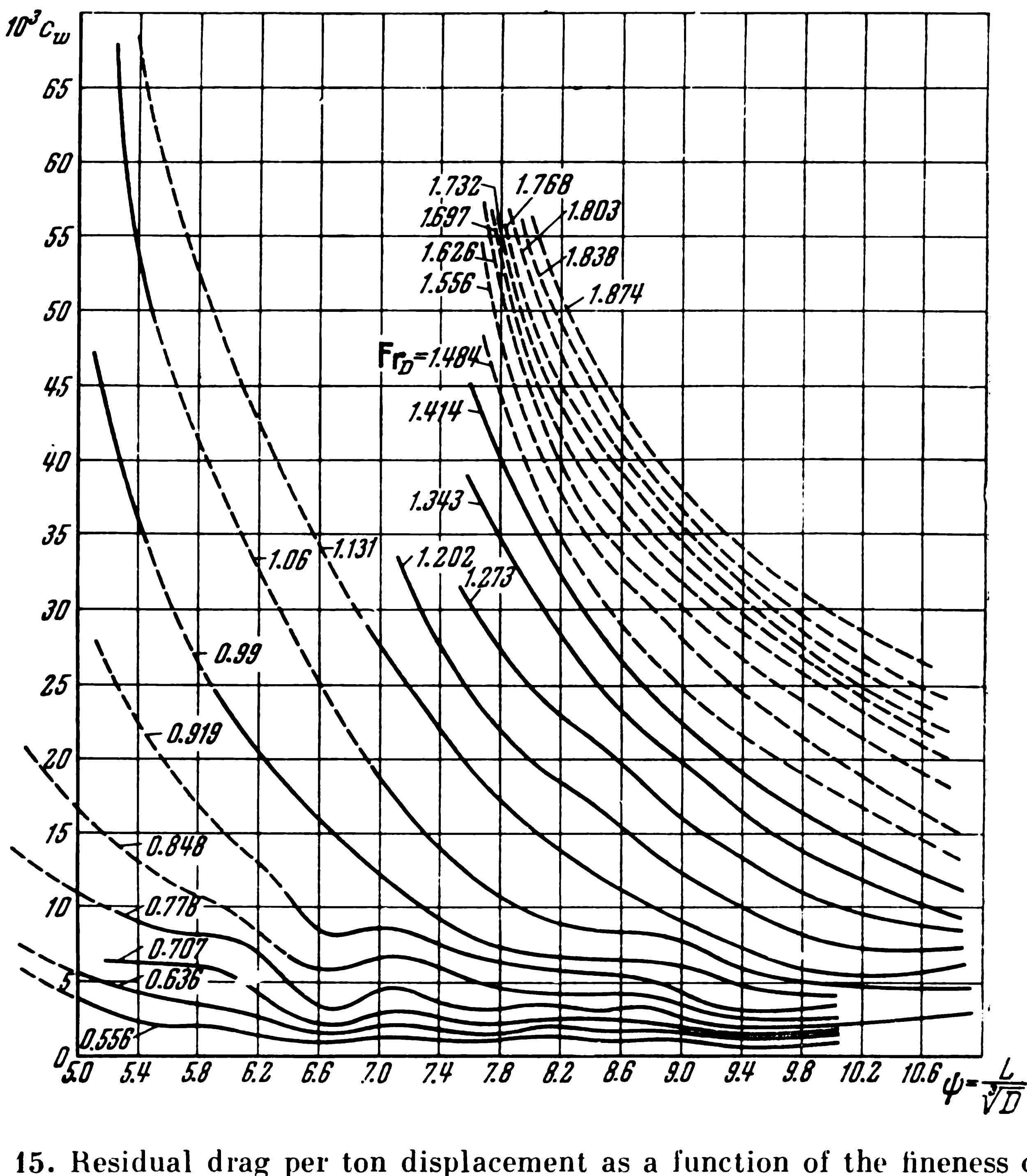

Fig. 15. Residual drag per ton displacement as a function of the fineness coefficient ψ and the Froude number Fr_D corresponding to the displacement.

power E needed at the propeller shaft. The power E or the number of revolutions n can be taken as the characteristic parameter instead of the velocity; in this case, the quantity to be determined becomes the velocity.

The product $W''v$ defines the effective power expended in propelling the ship. This power is always less than the power E delivered to the

propeller shaft since part of the power E is consumed in the additional perturbation of water by the propeller.

The ratio

$$\frac{W'v}{E} = \eta$$

is called the propulsive coefficient. The quantity η characterizes the hull efficiency, the efficiency of the propeller, and its operation in interaction with the hull. The best ships are characterized by large values of the propulsive coefficient.

The propulsive coefficient for ships with a given shape, a given coefficient ψ, a given propeller with definite location relative to the hull can be considered, for a constant ratio d/L (d is the propeller diameter), a function of the Froude and Reynolds numbers or a function of the propeller advance ratio $v/(nd)$ and of the Reynolds number; when the Reynolds number varies slightly, its effect is negligible. When the hull shape and the geometric data on the propeller vary, the value of η will depend on the parameters determining the shape of the hull and the propeller. These influences are felt in certain cases through a variation of the hull drag which is independent of the propeller operation; in other cases, through propeller characteristics which are independent of the hull shape. Finally, situations can arise in which the value of the propulsive coefficient is related to the hull-propeller interaction.

A tendency to construct large ships is observed in modern shipbuilding. Let us demonstrate arguments in favour of this tendency. Formula (9.2) can be rewritten in an alternative form

$$W = [c_f(\mathsf{Re}) + c'_w(\psi, \mathsf{Fr})]\,\rho\frac{Sv^2}{2} \tag{9.4}$$

where

$$c'_w = 2c_w\frac{D}{LS}\frac{1}{\mathsf{Fr}^2}$$

Take two geometrically similar ships with the displacements proportional to the cube of the linear dimensions:

$$\frac{D_1}{L_1^3} = \frac{D_2}{L_2^3}$$

which is equivalent to a natural assumption about the similarity of the submerged parts. Evidently, the wetted area S is proportional to the square of the linear dimensions.

Let L_1 and L_2 be the corresponding characteristic lengths, where $L_2 > L_1$. If the velocities are identical, we have

$$\mathsf{Re}_2 > \mathsf{Re}_1 \quad \text{and} \quad \mathsf{Fr}_2 < \mathsf{Fr}_1$$

It is known from experiment that the coefficient c_f decreases as the Reynolds number increases (see formula (9.3)) and that the coeffi-

cient c'_w also decreases with the Froude number (at least, in the range of small Froude number values $\mathsf{Fr} < 0.5$ common in practice). A typical dependence of c'_w on the Froude number is shown in Fig. 16.

If the motion occurs at the same velocity, then the drag ratio W_1/W_2 equals the ratio of the powers, or the ratio of fuel consumption per unit time if the efficiencies are identical. The transportable load is proportional to the displacement, i.e. to the cube of the linear dimensions. The cost of transporting one ton is defined by the ratio of the weight of the fuel consumed to the weight of the transported load. At the identical velocities the ratio of the costs Q_2 and Q_1 of transporting one ton by a distance of one kilometre is given by the following formula:

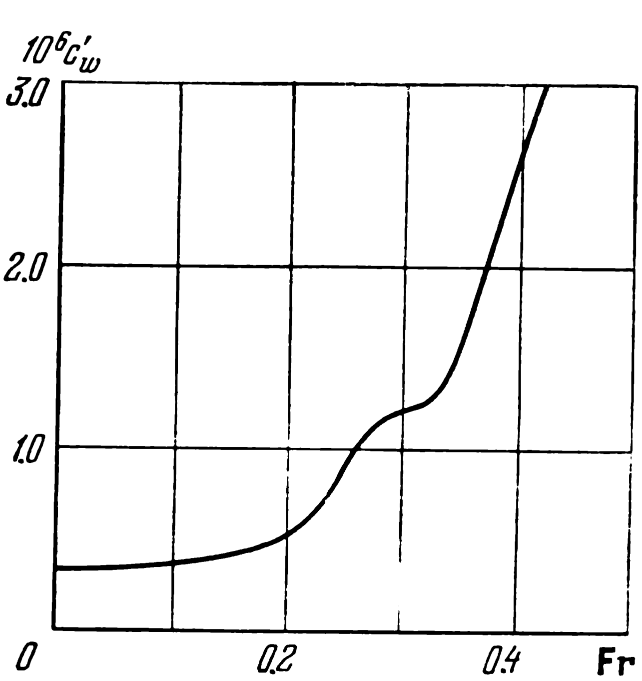

Fig. 16. Typical curve for the residual drag coefficient c'_w as a function of the Froude number Fr.

$$\frac{Q_2}{Q_1} = \frac{W_2 L_1^3}{W_1 L_2^3}$$

The last ratio is one of the most important elements in the economics of ship transportation. Using formula (9.4), we can write

$$\frac{Q_2}{Q_1} = \frac{c_f(\mathsf{Re}_2) + c'_w(\mathsf{Fr}_2)}{c_f(\mathsf{Re}_1) + c'_w(\mathsf{Fr}_1)} \cdot \frac{L_1}{L_2} = \varkappa \frac{L_1}{L_2} \tag{9.5}$$

where $\varkappa$ is a quantity less than unity and decreasing as the ratio L_2/L_1 increases.

Formula (9.5) shows that the quantity Q_2 decreases more rapidly than the first inverse power of the ship dimensions. The ratio of powers increases more slowly, for the same velocities, than the ratio of lengths squared.

By analogous reasoning, it can be shown that if the power increases in proportion to the cube of the linear dimensions, then the velocity is increased while the sailing time and the cost of transporting a ton of load per kilometre decrease.

The above considerations can be applied not only to ships moving over the surface of water but also to airplanes since the air drag increases slower than in proportion to the square of the linear dimensions at a fixed flight speed while the airplane weight and the useful load increase approximately in proportion to the cube of the linear dimensions. The fuel capacity and the range of airplanes therefore increase with their size. This explains the increase in the size and weight of aircraft intended for long-range flights.

In addition, the question of the suitable dimensions for aircraft engines, hydraulic machines, etc. is of great practical interest.

The ratio of the weight of a jet engine to the thrust it develops (weight-thrust ratio) is a most important characteristic of the engine for its use in airplanes.

It is more advantageous to use several engines of small size in order to obtain a given thrust than to use one engine of large size since the weight-thrust ratio of the engine increases with its size: the thrust increases proportionally to the square of the linear dimensions while the engine weight is approximately proportional to the cube of the linear dimensions.

Hence, in terms of the weight-thrust ratio of the engine, the use of expensive materials, and maintenance operations, it is more advantageous to construct several small engines than one large. These arguments become less valid for engines of very small size since mechanical similarity is lost when the size decreases sharply; hence, the thrust and useful power decrease very rapidly.

Besides these general qualitative considerations of similarity theory, the choice of the suitable size of engines and of hydraulic machines is also limited by the properties and dimensions of the auxiliary automatic mechanisms, by economic, technological, structural, and certain other requirements that must be taken into account in making a final decision.

The optimization of the dimensions of aircraft engines, hydraulic turbines, and many other machines must be analysed and studied thoroughly. Similarity considerations are of great value in this analysis.

§ 10. Planing over the Water Surface

Hydroplaning, or simply planing, can be visualized as sliding over the water surface. The supporting force during planing is specified almost entirely by the dynamic reaction of water. When displacing ships move, the supporting force is produced, just as at rest, by the buoyancy force due to the increase of the hydrostatic pressure with depth.

The planing principle is used in high-speed ships such as modern high-speed torpedo cutters. The run of a seaplane at take-off and the run after landing are accompanied by planing.

The planing of a ship of given shape can occur for different orientations of the ship relative to the water surface. The ship's orientation relative to water is of key importance.

The number of parameters defining the motion of a planing boat (glider) or seaplane of given geometric shape is larger than in the case, analysed above, of the motion of a ship which displaces water. In planing, besides the draft or the wetted area, the trim angle θ (the angle a certain direction fixed in the ship makes with the horizontal) must also be given. The loading Δ on water, the position of

the centre of gravity of the ship, and the moment of the external forces (say, aerodynamic, but never hydrodynamic forces) about the centre of gravity can be given instead of the draft and the trim angle θ. In practice, it is convenient to select the load on water and the trim angle as the decisive quantities.

If we formulate the planing problem in the same way as the problem of motion of a ship which displaces water, we find that steady planing of a ship of given geometric shape is determined by the following system of parameters:

$$B, \ \Delta, \ \theta, \ v, \ \rho, \ g, \ \mu$$

The set of dynamically similar motions and all dimensionless combinations formed from various mechanical quantities are defined by the values of the dimensionless parameters

$$\theta, \quad \frac{\Delta}{\rho g B^3} = C_\Delta, \quad \frac{vB\rho}{\mu} = \mathsf{Re}, \quad \frac{v}{\sqrt{gB}} = \mathsf{Fr} \tag{10.1}$$

By comparison with the motion of ships which displace water, the trim angle θ, which can have different values in the motions being compared, must be introduced. The difference in the trim angles for ships which displace water can be neglected in all cases of practical interest.

The effect of weight of water is expressed by means of parameters containing the acceleration due to gravity. The two parameters containing g in (10.1) are C_Δ and Fr. This system can be replaced by the system

$$\theta, \quad C_B = \frac{2\Delta}{\rho B^2 v^2} = \frac{2C_\Delta}{\mathsf{Fr}^2}, \quad \mathsf{Re}, \quad \mathsf{Fr} \tag{10.2}$$

into which the acceleration due to gravity enters only via the Froude number $\mathsf{Fr} = v/\sqrt{gB}$.

The planing phenomenon has a definite impulsive character. In front of a planing boat, water is practically at rest, and then water is set into motion by the approaching bottom of the boat in a short interval of time. This justifies the assumption that the inertial forces are the main forces compared with which the forces due to the weight of water particles are small and can be neglected.

The assumption that the weight of water particles can be neglected is equivalent to the assumption that the parameter g and, therefore, the Froude number can be neglected, in the system of parameters θ, C_B, Re, and Fr.

The unimportance of the effect of the weight of water on a number of fundamental characteristics of motion in the planing mode (large Froude number) has been established theoretically [17-19]. This is confirmed by a great deal of experimental material [20, 21]. Conse-

quently, the Froude similarity law is inadequate for modelling of purely planing motions.

It is incorrect to say that in the system of parameters θ, C_Δ, Re, and Fr the Froude number Fr and the coefficient C_Δ containing the acceleration due to gravity g can be neglected in the absence of the effect of the fluid weight. The parameters C_Δ and Fr form the combination

$$\frac{2C_\Delta}{\mathsf{Fr}^2} = C_B$$

that does not contain g. The parameter C_B can play an important part in a number of problems in which the weight of water and the parameter g are totally inessential.

The dimensionless quantities, independent of the Froude number and the Reynolds number, can depend on the load Δ and on the velocity v only through the combination

$$\frac{2\Delta}{\rho B^2 v^2} = C_B$$

Consequently, an investigation of the effect of the velocity can be replaced by an investigation of the effect of the load, and conversely. This result is of great value when the velocities attainable are inadequate.

Using this, we can obtain experimentally a number of results which are valid for cases that cannot be covered by direct experiment.

The viscosity of water is appreciable only directly at the bottom of the ship (boundary layer); consequently, the Reynolds number does not substantially influence the pressure distribution, the moment of the hydrodynamic forces, the shape of a wetted surface, etc. The effect of the viscosity of water on the damping of disturbances is only felt at very large distances from the ship.

The drag force depends essentially on the frictional force at the ship bottom surface; consequently, viscosity and the Reynolds number influence the dimensionless drag coefficients.

The hydrodynamic characteristics of steady planing strongly depend on the geometric shape of the ship hull, as well as on the mechanical parameters mentioned.

The difference in the nature of the supporting forces causes a distinct difference in the shape of planing ships from the shape of displacing ships. The outlines of the hull of planing ships are characterized by the flat-bottomed shape of the hull, by the abruptly drawn chines, and by the presence of transverse steps on the bottom of the ship. The flat-bottomed shape is necessary to absorb the large vertical forces on a small wetted surface. During planing the sharp chines and steps cause the breakaway of jets of water; consequently, the side surface of the ship and a significant part of the lower section

of the bottom are not wetted by water, which decreases the friction drag.

The various peculiarities of the bottom surface can be characterized by certain dimensionless parameters. The effects of the change of these parameters can be determined by systematically testing a series of profiles.

In the problem of the planing plate shaped like a plane wedge, we encounter a very interesting property, closely related to mechanical similarity and dimensional analysis. Consider a plane-keeled prismatic plate planing over the surface of water. We suppose that the keel of the plate possesses a vertical plane of symmetry and that the motion is parallel to this plane. The rear part of the plate is a plane perpendicular to the plane of symmetry.

Let us consider the case when the plate length and the width of the sidepiece of the wedge are large enough so that the boundaries of the wetted surface for comparable motions are not related to the structural width and length of the plate. The geometrical width and length of the plate can be assumed to be infinite for all comparable motions. The geometric shape of the plate is determined completely by the angle between the sidepieces $\pi - 2\beta$ (β is the careening angle) and by the angle between the keel and the plane surface. These angles can be taken as the geometric shape parameters. For simplicity, we shall analyse a class of motions in which these angles are fixed.

It is not difficult to see that the number of characteristic parameters is reduced in this case since the linear dimension characterizing the plate is absent.

The steady planing of a plane-keeled plate with an incompletely wetted width is defined by the parameters

$$\Delta, \ \theta, \ v, \ \rho, \ g, \ \mu$$

The wetted width of the plate along the rear part and the wetted length along the keel are defined completely by the parameters mentioned.

The class of similar motions and the mode of motion are characterized by the three dimensionless quantities:

$$\theta, \quad \frac{v}{\sqrt{g\sqrt[3]{\Delta/(\rho g)}}} = \mathsf{Fr}_1, \quad \frac{\rho v \sqrt[3]{\Delta/(\rho g)}}{\mu} = \mathsf{Re}_1 \qquad (10.3)$$

The numbers Fr_1 and Re_1 can be considered the Froude and Reynolds numbers associated with the load.

The Reynolds number for the planing mode (larger Fr_1) only affects quantities which depend on the nature of fluid motion in a boundary layer. In particular, it can be assumed that the wetted length l along the keel is independent of viscosity, i.e.

$$l = f(\theta, \ \Delta, \ v, \ \rho, \ g) \qquad (10.4)$$

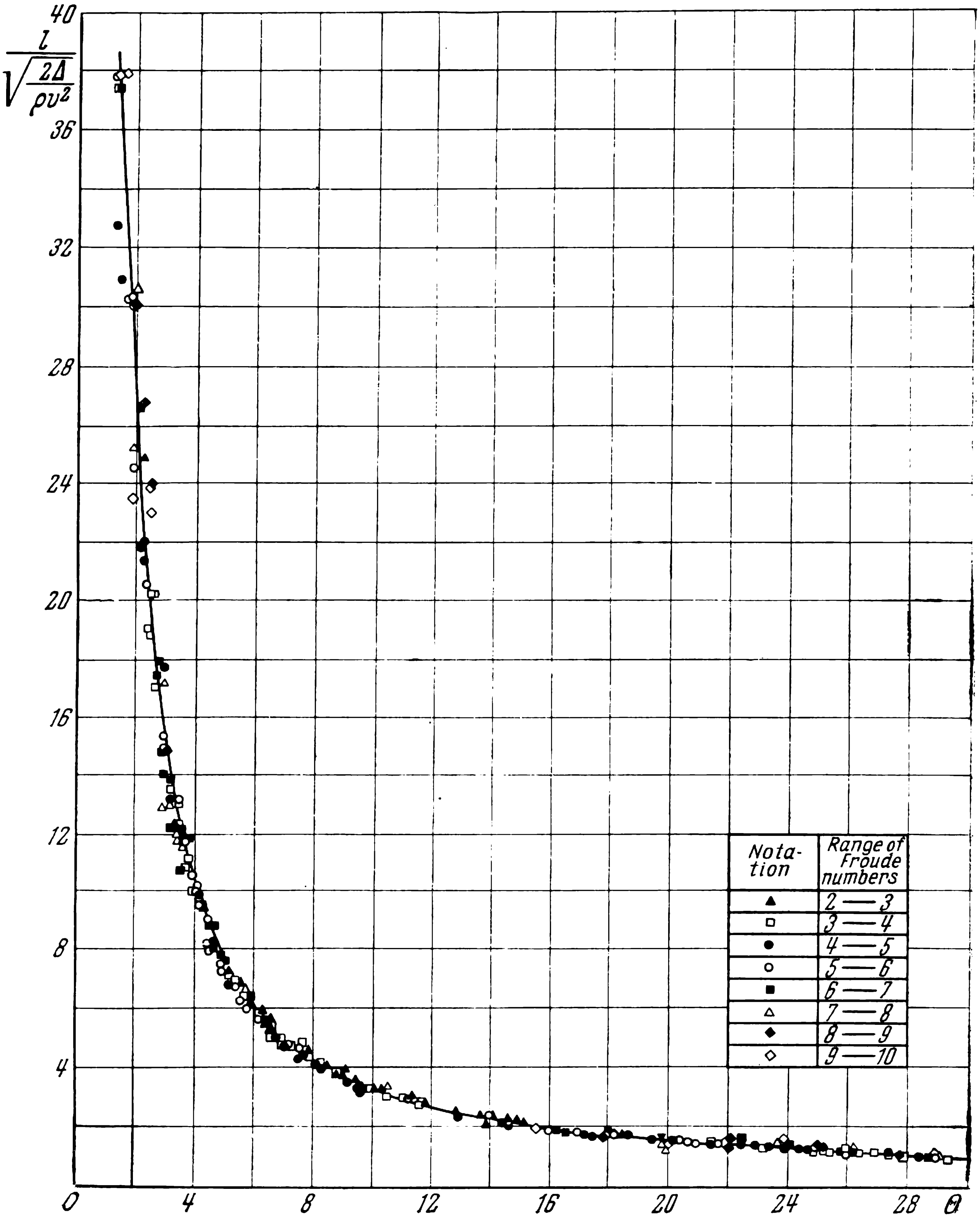

Fig. 17. Gliding of a careening plate.

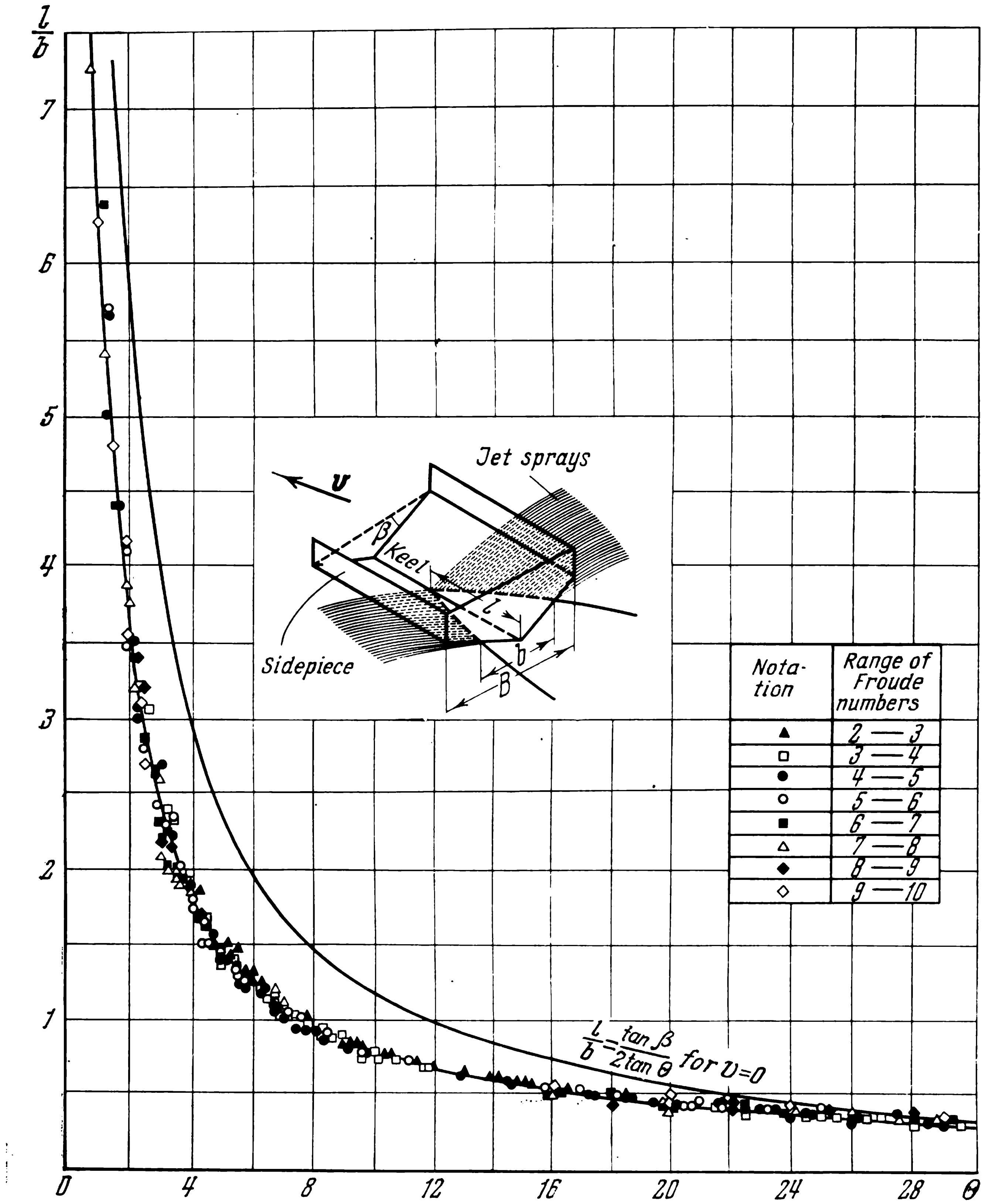

Fig. 18. Experimental data showing that the effect of the weight of water is not important for the Froude numbers $Fr_1 > 2$.

This relation can be written in dimensionless form,

$$l = \sqrt{\frac{\Delta}{\rho v^2}}\, f(\theta, \mathrm{Fr}_1) \tag{10.5}$$

If it is assumed that the weight does not influence the magnitude of l, i.e. that the acceleration g in (10.5) is not essential, we obtain

$$l = \sqrt{\frac{\Delta}{\rho v^2}}\, f(\theta) \tag{10.6}$$

Under these assumptions, it is evident that all the linear dimensions (for example, the wetted width along the rear end, etc.) are

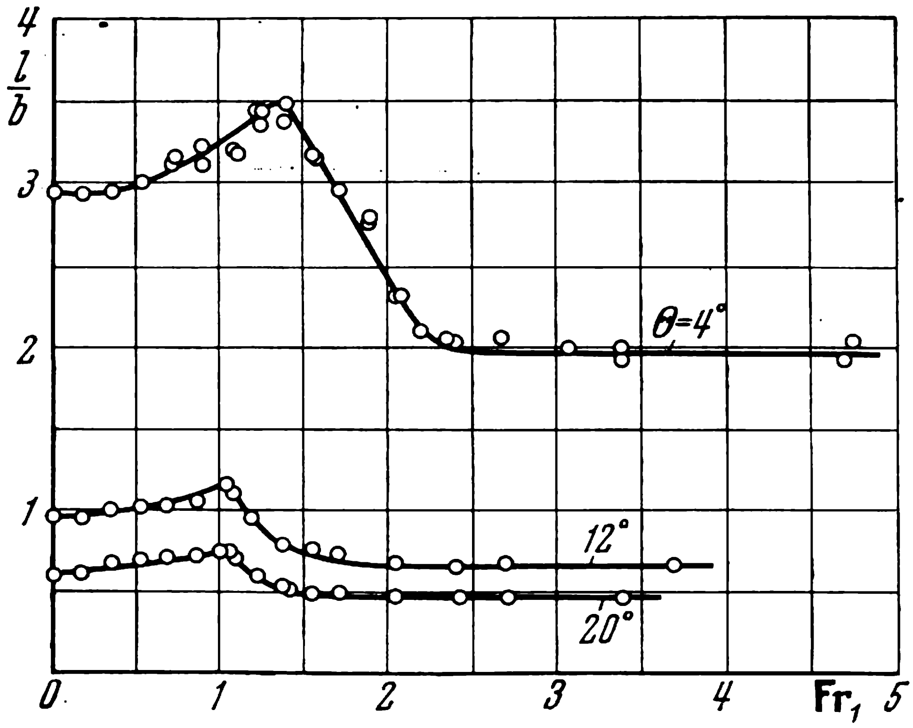

Fig. 19. Gliding of careening plates. The effect of weight is important only for the Froude numbers $\mathrm{Fr}_1 < 2$.

proportional to $\sqrt{\Delta/(\rho v^2)}$. The wetted surfaces will be geometrically similar for fixed trim angle θ but with different loads Δ on water and at different planing velocities v.

Experiments confirm the validity of these conclusions for large values of the Froude number when $\mathrm{Fr}_1 > 2$ [22] (Figs. 17, 18, and 19).

We took the load Δ on water and the velocity of motion v as natural parameters in the system of characteristic parameters precisely because these quantities are given beforehand in experiments while the rest are measured. The wetted length l along the keel or the wetted width b along the rear end can be taken as characteristic parameters instead of the load Δ. For example, the following system of dimensional characteristic parameters can be taken:

$$\theta,\ l,\ v,\ \rho,\ g,\ \mu$$

the corresponding system of dimensionless parameters is

$$\theta, \quad \frac{v}{\sqrt{gl}}, \quad \frac{\rho v l}{\mu}$$

In this case, relations (10.4) to (10.6) can be considered as relations determining the load Δ on water.

§ 11. Impact on Water

The phenomenon of impact on water is encountered in a number of cases; in particular, when a seaplane lands on water. Of particular interest in the study of this phenomenon are the explanation of the properties of the reaction force of water and the investigation of water ricochet.

Many readers have probably observed the skimming of flat stones on water. If the stones are thrown at velocities having a large horizontal component and with rotation which guarantees the conservation of a small angle between the plane of the stone and the horizontal, they easily recoil upward upon contact with water and may bounce several times. Obviously, the horizontal velocity plays a fundamental part in this phenomenon of ricochetting. A flat heavy stone cannot recoil from water if the horizontal velocity is negligible. Multiple ricochets testify to the small loss in horizontal velocity during contact with water. The ricochetting of shells is another well-known phenomenon. For example, a circular cannonball of 0.16 m in diameter at an initial velocity of 455 m/s can perform more than 22 ricochets off water [23].

The ricochet firing is sometimes used deliberately in artillery.

The phenomenon of water ricochet can occur when a seaplane lands on water. Water ricochet of an airplane is a cause of anxiety in the air transport industry and is considered to be an extremely undesirable phenomenon.

Let us use dimensional analysis and similarity theory to analyse the problem of impact on water as applied to seaplane landings and to experiments with plates serving as simulated models of boats and airplane floats.

We suppose that the geometric shape of a moving body is fixed. The scale, however, which can be chosen as the value of linear dimension, is kept variable. The width of the boat is naturally taken as the characteristic linear dimension. For simplicity, we limit ourselves to the case when the body is absolutely rigid. The differences in aerodynamic characteristic can be reflected strongly in the course of ricochetting. An experimental investigation of the problem can start with an analysis of the motion of bodies of identical geometric shape and, therefore, with identical aerodynamic properties.

Let us assume that the body has a longitudinal plane of geometric and dynamic symmetry and that the body moves in a longitudinal direction parallel to the plane of symmetry. Furthermore, we assume that the motion of the body is unsteady with two degrees of freedom: a vertical displacement of the centre of gravity and a rotation about the centre of gravity can occur. We suppose that the horizontal velocity of the centre of gravity is fixed. Actually, a decrease in the horizontal velocity occurs because of the drag; however, this decrease can be neglected owing to the short duration of the most interesting processes. (It is not essential to assume that the horizontal velocity is constant in order to apply dimensional analysis to the general formulation of the water impact problem. The necessity to fix the horizontal velocity arises in modelling this phenomenon. The constancy of the horizontal component can be guaranteed under laboratory conditions by the motion of a tow-cart; the model centre of gravity can slide along the vertical guide fastened to the cart that moves at a given constant velocity. The frictional forces arising in the guiding instrumentation can be made negligibly small.)

We shall take account of fluid inertia, viscosity, and weight in the general formulation of the problem. We neglect compressibility and capillarity. The wave motion of water can exert a substantial effect on the phenomenon being studied, but we shall assume that prior to the impact the water adjacent to the body was at rest.

From the assumptions made, it follows that the motion of the body-water system is defined by the following system of parameters which can be given arbitrarily within certain limits:

(1) The scale parameter: the boat width B (the plate width in the case of a model).

(2) The kinematic parameters: the instant of time t under consideration (the initial instant $t = 0$ corresponds to the moment the body touches the water surface), the horizontal velocity U and the initial vertical velocity v_0, the initial trim angle (the angle of attack) θ_0, and the initial angular velocity Ω_0 (we assume that the initial state of motion at the moment of landing is completely defined in practice by the parameters U, v_0, θ_0, and Ω_0).

(3) The dynamic parameters of the body: the coordinates of the centre of gravity ξ and η in a certain coordinate system fixed to the body, the moment of inertia J relative to the transverse axis passing through the centre of gravity, the mass m, and the vertical component A of the given external forces ($A = mg$ for a free body). Under laboratory conditions, the quantities A and mg can be made independent by using an artificial counterweight. (We assume that the external additionally assigned forces are applied at the centre of gravity. The total aerodynamic forces and moments can be considered to be the quantities defined by the shape and motion of the body.)

(4) The physical constants: the acceleration due to gravity g, the density ρ, and the viscosity μ of water.

All the dimensionless quantities related to our problem are functions of the following set of dimensionless parameters defining the mode and the state of motion:

$$\tau = \frac{Ut}{B}, \quad \frac{v_0}{U}, \quad \theta_0, \quad \frac{\Omega_0 B}{U}, \quad \frac{\xi}{B}, \quad \frac{\eta}{B}, \quad \frac{J}{\rho B^5}, \quad \frac{m}{\rho B^3},$$
$$C_B = \frac{2A}{\rho B^2 U^2}, \quad \mathrm{Fr} = \frac{U}{\sqrt{gB}}, \quad \mathrm{Re} = \frac{\rho U B}{\mu}$$

The effect of the Reynolds number Re is felt through the frictional forces depending on the viscosity of water, which are not generally large in comparison with the lift and which are usually directed approximately in a horizontal direction; however, in certain cases, the effect of frictional forces on the magnitude of the rotational moment is significant. But if a comparatively weak dependence of the frictional forces on the Reynolds number is taken into account, then, apparently, it is completely valid to neglect the effect of the Reynolds number on the characteristics of the vertical and angular motions and, in particular, on the phenomenon of water ricochet.

The effect of the Froude number Fr on the hydrodynamic forces, the shape of the wetted surface, etc. is related to the effect of the weight of water on the perturbed motion of the water near the body. At high horizontal speeds, the phenomenon is of an impact character; consequently, the water reaction forces can be considered to be independent of the Froude number. Also it should be remembered that a sufficiently large value of the Froude number, at which the effect of this number begins to be insignificant, depends on the character of the mechanical quantities under consideration and is related to the values of other characteristic parameters.

If $A = mg$, then the parameters $m/(\rho B^3)$, C_B, and Fr become dependent. In this case, it is sufficient to retain only the two parameters $m/(\rho B^3)$ and C_B since the effect of the weight of a model and water is taken into account simultaneously by the C_B coefficient.

Let us denote the vertical component of the total hydrodynamic forces by Y. From the above, we deduce the following formula:

$$\frac{Y}{A} = f\left(\tau, \frac{v_0}{U}, \theta_0, \frac{\Omega_0 B}{U}, \frac{\xi}{B}, \frac{\eta}{B}, \frac{J}{\rho B^5}, \frac{m}{\rho B^3}, C_B\right) \quad (11.1)$$

The parameters ξ/B, η/B, $J/(\rho B^5)$, and $m/(\rho B^3)$ are constant for the motion of a specific model. If the trim angle is fixed and only the vertical velocity is variable, then we have the motion with only one degree of freedom, namely, the vertical forward motion. In this case, $\Omega_0 = 0$ and the parameters ξ, η, and J are insignificant;

formula (11.1) becomes

$$\frac{Y}{A}=f\left(\tau,\ \frac{v_0}{U},\ \theta,\ \frac{m}{\rho B^3},\ C_B\right) \tag{11.2}$$

The trim angle is retained in this formula as a constant which can vary in different experiments.

The parameter τ is defined if we consider the maximum value of the ratio Y/A in different experiments, or, in general, the value of Y/A for certain characteristic moments of time

$$\frac{Y_{\max}}{A}=f^*\left(\frac{v_0}{U},\ \theta,\ \frac{m}{\rho B^3},\ C_B\right) \tag{11.3}$$

It is evident that the average values of all the quantities during the period of contact with water, the dimensionless time $\tau_1 = Ut_1/B$, the dimensionless maximum depth of submersion $h_{\max}/B$, etc. are not related to the parameter τ.

The whole set of motions can be subdivided into two parts corresponding to the cases when the body does or does not emerge from water (the presence or absence of ricochets). The boundary between these two modes is defined by the relation

$$\Phi\left(\frac{v_0}{U},\ \theta_0,\ \frac{\Omega_0 B}{U},\ \frac{\xi}{B},\ \frac{\eta}{B},\ \frac{J}{\rho B^5},\ \frac{m}{\rho B^3},\ C_B\right)=0 \tag{11.4}$$

and in the case of forward motion with one degree of freedom, by the relation

$$\Phi\left(\frac{v_0}{U},\ \theta,\ \frac{m}{\rho B^3},\ C_B\right)=0 \tag{11.5}$$

Some experimental data are available on the form of the function Φ for a plane plate landing on water [24].

While considering the problem of steady planing of a keeled plate, we indicated the possibility of decreasing the number of characteristic parameters in the case when the wetted surface does not depend on the dimensions of the plate, as a result of which the parameter B is neglected.

In the problem of a wedge (with a fixed angle) of the kind described in the preceding section making impact on water, the following formulas hold instead of (11.3) and (11.5):

$$\frac{Y_{\max}}{A}=f^*\left(\frac{v_0}{U},\ \theta,\ \frac{2A}{\rho^{1/3}m^{2/3}U^2}\right) \tag{11.6}$$

and

$$\Phi\left(\frac{v_0}{U},\ \theta,\ \frac{2A}{\rho^{1/3}m^{2/3}U^2}\right)=0 \tag{11.7}$$

We can replace the parameters v_0/U and $2A/(\rho^{1/3}m^{2/3}U^2)$ by the equivalent parameters

$$\frac{v_0}{\sqrt{\frac{A}{m}\sqrt[3]{\frac{m}{\rho}}}} \quad \text{and} \quad \frac{U}{\sqrt{\frac{A}{m}\sqrt[3]{\frac{m}{\rho}}}}$$

that are symmetric and represent the effect of the vertical and horizontal velocities separately, for constant A and m.

Above, we noted the system of characteristic parameters and the form of certain important relations. Some of these parameters are not essential in a number of cases; this aspect is clarified by means of special investigations outside the scope of dimensional analysis and similarity theory.

The phenomenon of ricochetting on the water surface is closely related to the phenomenon of the longitudinal instability of planing. We encounter the phenomenon of the longitudinal instability of planing in actual motion of seaplanes and planing boats and in experiments with models. It is well known that unstable modes of motion exist for every seaplane and for every model. Strong longitudinal oscillations in these modes are extremely unfavourable and dangerous. Just as in the problem of landing on water, the investigation of the planing instability phenomenon is complicated by a large number of parameters whose effect must be clarified.

It is not difficult to see that a system of dimensionless parameters defining planing instability is obtained from the system of parameters determining the water impact phenomenon if we put $v_0 = \Omega_0 = 0$. A similar concept of the boundary separating the stable and unstable planing modes corresponds to the concept of the ricochet boundary.

If the effect of the weight of water is negligible, then the stability boundaries depend on the load and on the planing velocity only through the coefficient $C_B = 2A/(\rho B^2 U^2)$. This is in good agreement with experimental results for a number of modes important in practical situations [25].

As in the water impact problem, the number of parameters defining the planing stability of a plane-keeled plate of indefinite width is decreased [26].

Let us now concentrate on the problem of vertical entry into water.

The phenomenon of vertical water impact when a body is moving forward (if the body is not symmetric, the vertical forward motion can be accomplished by using special guides) is defined by the following parameters:

$$t,\ v_0,\ m,\ B,\ A,\ g,\ \text{and}\ \rho$$

We take the following four quantities as dimensionless parameters defining the mode and the state of motion:

$$\frac{tv_0}{\sqrt[3]{\frac{m}{\rho}}}, \quad \frac{m}{\rho B^3}, \quad \frac{v_0}{\sqrt{g\sqrt[3]{\frac{m}{\rho}}}}, \quad \frac{v_0}{\sqrt{\frac{A}{m}\sqrt[3]{\frac{m}{\rho}}}} \tag{11.8}$$

If $A = mg$, only the three parameters remain.

It is evident that the dimensionless quantities adopted as characteristic moments of time (maximum or average values in time) are determined by only two parameters for $A = mg$:

$$\frac{m}{\rho B^3}, \quad \frac{v_0}{\sqrt{g\sqrt[3]{\frac{m}{\rho}}}}$$

For example, the expressions for the maximum impact and for the momentum of water acting upon a body in a certain characteristic time interval are

$$P_{\max} = f_1\left(\frac{m}{\rho B^3}, \quad \frac{v_0}{\sqrt{g\sqrt{\frac{m}{\rho}}}}\right)\rho\left(\frac{m}{\rho}\right)^{2/3} v_0^2 \tag{11.9}$$

and

$$I = f_2\left(\frac{m}{\rho B^3}, \quad \frac{v_0}{\sqrt{g\sqrt[3]{\frac{m}{\rho}}}}\right) mv_0 \tag{11.10}$$

If the velocity v_0 on contact is large and the shape of the wetted surface of a body is almost a horizontal plane, then the process of submersion into water has a definite impact character. In this case, the weight of water and the body weight are not essential. Consequently, for entry into water when the quantity

$$\frac{v_0}{\sqrt{g\sqrt[3]{\frac{m}{\rho}}}}$$

is large enough, we must have

$$P_{\max} = f_1\left(\frac{m}{\rho B^3}\right)\rho^{1/3} m^{2/3} v_0^2 \tag{11.11}$$

$$I = f_2\left(\frac{m}{\rho B^3}\right) mv_0 \tag{11.12}$$

It is evident in the case of constant v_0 and m that the fluid reaction will increase as the body dimensions (the parameter B) increase; consequently, the functions $f_1(m/(\rho B^3))$ and $f_2(m/(\rho B^3))$ must increase when the parameter $m/(\rho B^3)$ tends to zero. Formulas (11.11) and

(11.12) show that the maximum force is proportional to the square of the velocity but the momentum is proportional to the first power of the incident velocity [27].

The parameter $m/(\rho B^3)$ is eliminated in the problem of cone entry into a water surface (the conical section by a horizontal plane can be arbitrary) since there is no preassigned characteristic linear dimension in this case.

Thus in the problem of cone entry into water we obtain, instead of (11.11) and (11.12), the relations

$$P_{\max} = c_1 \rho^{1/3} m^{2/3} v_0^2 \tag{11.13}$$

$$I = c_2 m v_0 \tag{11.14}$$

These formulas determine the variation of $P_{\max}$ and I with mass. The constants c_1 and c_2 depend on the cone shape. It is interesting to note that the effect of the mass is independent of the cone shape in (11.13) and (11.14). The cone shape only influences the values of the constants c_1 and c_2.

However, the general conclusion that the maximum force is proportional to $m^{2/3}$ for bodies of any shape would be incorrect, in general. In fact, let us consider a long plane wedge incident on water at a small careening angle. Let the plane of symmetry of the wedge be vertical and let the incident velocity be large. Neglecting the weight of water and the wedge, we obtain the following system of characteristic parameters:

$$t, \ v_0, \ \frac{m}{L} = m_1, \ L, \ \rho$$

where L is the length of the wedge along the keel, and m_1 is the wedge mass per unit length.

If L is very large, then we can consider the limiting case $L = \infty$. The linear dimension is eliminated in the limiting case of a plane infinitely long wedge when the wetted surface does not reach the wedge edges; consequently, the phenomenon is defined by only the four dimensional quantities:

$$t, \ v_0, \ m_1, \ \rho$$

All the dimensionless characteristics are defined by the one dimensionless quantity

$$\frac{tv_0}{\sqrt{\dfrac{m}{\rho L}}}$$

The maximum and average values of the dimensionless mechanical characteristics will be dimensionless constants.

Hence, the following formula will be correct for the maximum impact force in this case:

$$\frac{P_{\max}}{L} = c_3 \sqrt{\rho m_1}\, v_0^2$$

or

$$P_{\max} = c_3 \sqrt{\rho m L}\, v_0^2$$

Therefore, the maximum force for a very elongated body is proportional to the square root of the body mass. The constant c_3 depends on the careening and it can be considered a function of the careening angle β.

It is not difficult to see that c_3 increases as the careening angle decreases. For small careening angles, it is possible to put $c_3 = c_4/\beta$, where c_4 can be considered independent of the careening angle for small β. In the general case of a finite wedge, the following formula can be given for c_3:

$$c_3 = \frac{f\left(\frac{m}{\rho L^3},\ \beta\right)}{\beta}$$

It is convenient to express the experimental results in terms of the function $f\,(m/(\rho L^3),\ \beta)$ since this function will vary slightly for small β and for very elongated wedges or small m.

§ 12. Entry of a Cone and a Wedge at Constant Speed into a Fluid

Let us consider the problem of the unsteady motion of an incompressible fluid caused by the entry of a solid body in the shape of a cone or a wedge into the fluid. The shape of the cone in the three-dimensional case and the shape of the plane wedge of infinite span in the plane case are of interest in that their surfaces are fixed completely by the single requirement of geometrical similarity. The set of geometrically similar cones reduces to a single unique cone. The surface of the cone and the surface of the plane wedge are defined completely by dimensionless geometric quantities.

We assume that the fluid occupies the whole lower half-space bounded by the horizontal plane and that the properties of the fluid weight and viscosity can be neglected. Therefore, we shall assume that the fluid is incompressible, homogeneous, and ideal.

Let $t = 0$ be the initial instant at which the body makes contact with the fluid at rest. The body entering into the fluid moves forward with a velocity v which is constant in magnitude and direction.

We now assume that the pressure p on the free surface has the constant value p_0. Since the fluid is incompressible, the value p_0 on the free surface cannot influence the perturbed fluid motion. We

can consider the difference $p - p_0$ instead of the pressure p; in this case, the parameter p_0 is not essential. Consequently, the mechanical properties of the fluid are determined by a single parameter, the density ρ.

It follows from the above that all the mechanical characteristics of the fluid motion are defined at each point by the following quantities:

$$\rho, \ t, \ v, \ \alpha, \ \beta, \ x, \ y, \ z$$

where α and β are the angles defining the direction of the velocity v relative to the body, and x, y, and z are the coordinates of the point under consideration either in a fixed coordinate system with the origin at the point of contact of the cone apex and the fluid level, or in a moving coordinate system which is fixed to the body with its origin at the cone apex. (We also assume, for the sake of simplicity, that the ratio of the ambient medium density above the liquid to the density of the liquid is fixed.)

It is evident that all the relevant dimensionless quantities defined by the parameters

$$\alpha, \ \beta, \ \frac{x}{vt}, \quad \frac{y}{vt}, \quad \frac{z}{vt}$$

in which the parameters $x/(vt)$, $y/(vt)$, and $z/(vt)$ can influence only the quantities dependent on the position of the point in the fluid. The total dimensionless parameters (for example, the total fluid reaction, etc.) or the parameters which are independent of position depend only on the angles α and β. If the direction of the velocity is fixed (for example, the velocity is vertical), then all the dimensionless total characteristics can be considered the absolute constants dependent only on the cone shape.

Let us denote the velocity potential of the perturbed fluid motion by $\varphi(x, y, z, t)$, where the cone velocity is given in magnitude and direction. Because the fluid motion is unsteady, the problem of determining the perturbed motion reduces to the determination of the velocity potential as a function of the four independent variables x, y, z, and t.

The four independent variables can easily be reduced to three by dimensional analysis.

Indeed, the quantity $\varphi/(v^2 t)$ is dimensionless, consequently, we have

$$\varphi = v^2 t f\left(\frac{x}{vt}, \quad \frac{y}{vt}, \quad \frac{z}{vt}\right) \tag{12.1}$$

The velocity of the fluid particles $\bar{\mathfrak{v}}$ is given by

$$\bar{\mathfrak{v}} = v\mathfrak{f}_1\left(\frac{x}{vt}, \quad \frac{y}{vt}, \quad \frac{z}{vt}\right) \tag{12.2}$$

The magnitudes of the total water reaction P and of the wetted area S are given by the formulas

$$P = c_1 \rho v^4 t^2, \quad S = c_2 v^2 t^2 \tag{12.3}$$

in which the coefficients c_1 and c_2 and the direction of the force P depend only on the cone shape and on the direction of the velocity of the cone motion. The first formula of (12.3) shows that the water reaction is proportional to the fluid density, to the fourth power of the velocity, and to the time squared. The wetted area is proportional to the velocity squared and to the time squared. Evidently, the two different states of the motion are dynamically similar.

The following formulas for the velocity potential and for the velocity distribution are correct in the two-dimensional case of entry of a plane wedge (the xy-plane is the plane of motion)

$$\varphi = v^2 t f\left(\frac{x}{vt}, \frac{y}{vt}\right), \quad \bar{\mathfrak{v}} = v \mathfrak{f}_1\left(\frac{x}{vt}, \frac{y}{vt}\right)$$

For the force per unit length of the wedge and for the wetted length along the wedge sidepiece, we have

$$P_1 = c_1' \rho v^3 t, \quad l = c_2' vt \tag{12.4}$$

It is seen from (12.3) and (12.4) that the relation between velocity and the water reaction when a body enters into water at a constant velocity is different for bodies of different shapes.

The constants c_1' and c_2' depend on the careening angle of the wedge, on the angles of inclination of the wedge plane of symmetry, and on the wedge velocity relative to the unperturbed level of the free surface.

Approximate theoretical solutions exist for the plane problem of vertical entry and for the penetration of a plate slightly inclined to the fluid level when the horizontal component of the plate velocity is large [28, 29].

§ 13. Small-Amplitude Waves on the Surface of an Incompressible Fluid

Considering the Cauchy-Poisson problem of waves on the surface of a heavy incompressible fluid, N. E. Kochin [30] used the reasoning of dimensional analysis and gave the solution of this classical problem a new elegant mathematical form.

Amplifying the reasoning of dimensional analysis, an entire class of new solutions of the wave problem can be found in explicit and simple form [31]. Kochin's solution is a particular case in the class of solutions obtained.

This method and the solutions found can be extended and generalized to the case of the three-dimensional problem.

The plane problem of potential waves of infinitely small amplitude on the surface of a heavy incompressible fluid occupying the whole lower half-space can be formulated as follows.

Let us take a Cartesian coordinate system; let the x-axis coincide with the unperturbed fluid level; let the y-axis be directed vertically upward. The velocity potential $\varphi(x, y, t)$ at $y < 0$ is a regular harmonic function, i.e.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{13.1}$$

when $y < 0$. Furthermore, we shall consider cases in which the fluid motion attenuates as the depth of submersion increases, i.e.

$$[\text{grad}\ \varphi] \to 0 \tag{13.2}$$

when $y \to -\infty$.

The condition of constant pressure on the free surface can be represented in linearized form

$$\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial y} = 0 \tag{13.3}$$

when $y = 0$ and $t \geqslant 0$, where g is the acceleration due to gravity. Certain requirements, in addition to conditions (13.1), (13.2), and (13.3), must be imposed in order to determine the velocity potential $\varphi(x, y, t)$.

To provide these additional data we assume that the shape of the free surface and the distribution of the pressure impulses at $t = 0$ are given.

The initial conditions can be formulated in linearized theory by starting from the relations

$$\zeta = -\frac{1}{g} \frac{\partial \varphi}{\partial t}\bigg|_{y=0}, \qquad \frac{p_t}{\rho} = -\varphi \tag{13.4}$$

where $\zeta(x, t)$ is the height of a point of the free surface above the unperturbed level, p_t is the pressure impulse, and ρ is the fluid density.

The initial conditions can be formulated as follows: we have at $t = 0$

$$-\frac{1}{g} \frac{\partial \varphi}{\partial t}\bigg|_{y=0} = f(x), \qquad \varphi\,|_{y=0} = F(x) \tag{13.5}$$

If the functions f and F are simultaneously neither zero nor infinite and if $f \neq kx$, then obviously these functions must depend on certain dimensional constants in addition to the variable x. Since the problem is formulated in kinematic terms, not more than two dimensional constants with independent dimensions entering into the functions f and F can exist.

We shall satisfy the Laplace equation if we put

$$\varphi(x, y) = \operatorname{Re} w(x + iy) \tag{13.6}$$

where $w(z, t)$ $(z = x + iy)$ is the characteristic function of the liquid flow. Furthermore, we assume that the function $w(z)$ is single-valued, finite, and regular when $t > 0$ and finite x and $y < 0$.

It follows from condition (13.2) that the derivative $\partial w/\partial z$ vanishes as $y \to -\infty$.

Boundary condition (13.3) can be written in the form

$$\operatorname{Re}\left(\frac{\partial^2 w}{\partial t^2} + ig\frac{\partial w}{\partial z}\right) = 0 \tag{13.7}$$

when $y = 0$. This condition permits the combination

$$G(z) = \frac{\partial^2 w}{\partial t^2} + ig\frac{\partial w}{\partial z}$$

to be continued into the upper half-plane; we then find that the function $G(z)$ is single-valued in the whole plane of the complex variable $z = x + iy$. It follows from the assumptions made about the general character of the fluid motion that the singular points of $G(z)$ lie on the real axis.

Let us look for solutions in which the characteristic function $w(z)$ depends linearly on the dimensional constants entering into the additional conditions determining the velocity potential; we will not specify the form of these conditions.

It follows from the linearity of the problem that it is sufficient to consider the case when we have only one dimensional constant a on which the characteristic function $w(z)$ depends linearly (the constant a may be complex). Let the dimensions of the constant a be represented by the formula

$$[a] = L^pT^q$$

From the assumptions made above, we deduce that the complete system of characteristic parameters is represented by the table

$$z = x + iy, \ t, \ g, \ a$$

Now, let us put

$$w = az^{\alpha}g^{\beta}\chi(z, t, g)$$

where the exponents α and β are selected so that χ is an abstract quantity. Since $[a] = L^pT^q$, then we must have

$$p + \alpha + \beta = 2, \quad q - 2\beta = -1$$

from which

$$\beta = \frac{1+q}{2}, \quad \alpha = \frac{3-2p-q}{2} \tag{13.8}$$

It follows from dimensional analysis that the function $\chi(z, t, g)$ depends only on the combination

$$\lambda = \frac{gt^2}{z}$$

that is

$$w = ag^{\beta} z^{\alpha} \chi(\lambda) \tag{13.9}$$

From (13.9) we obtain the following equations:

$$\frac{\partial w}{\partial t} = ag^{\beta} z^{\alpha} \chi'(\lambda) \frac{\partial \lambda}{\partial t}$$

$$\frac{\partial^2 w}{\partial t^2} = ag^{\beta} z^{\alpha} \left[\chi''(\lambda) \left(\frac{\partial \lambda}{\partial t} \right)^2 + \chi'(\lambda) \frac{\partial^2 \lambda}{\partial t^2} \right]$$

$$\frac{\partial w}{\partial z} = ag^{\beta} z^{\alpha} \left[\frac{\alpha}{z} \chi(\lambda) + \chi'(\lambda) \frac{\partial \lambda}{\partial z} \right]$$

and

$$G(z) = ag^{\beta} z^{\alpha} \left[\chi'' \left(\frac{\partial \lambda}{\partial t} \right)^2 + \chi' \left(\frac{\partial^2 \lambda}{\partial t^2} + ig \frac{\partial \lambda}{\partial z} \right) + \frac{ig\alpha}{z} \chi \right]$$

since

$$\frac{\partial \lambda}{\partial t} = 2\frac{\lambda}{t}, \quad \frac{\partial^2 \lambda}{\partial t^2} = 2\frac{\lambda}{t^2}, \quad \frac{\partial \lambda}{\partial z} = -\frac{\lambda}{z}$$

then

$$G = 4ag^{\beta+\alpha} t^{2\alpha-2} \lambda^{2-\alpha} \left[\chi'' + \left(\frac{1}{2\lambda} - \frac{i}{4} \right) \chi' + \frac{i\alpha}{4\lambda} \chi \right] \tag{13.10}$$

The function $G(\lambda)$ is a single-valued function of the complex variable λ in the whole plane. Singular points occur only on the real axis. Since the function $G(\lambda)$ is pure imaginary for real λ, then evidently the coefficients of the Laurent series are pure imaginary at any singular point of this function.

Let us denote:

$$a = a_0 e^{i\delta} \quad \text{and} \quad G = 4a_0 g^{\beta+\alpha} t^{2\alpha-2} G_1(\lambda)$$

We obtain from (13.10)

$$e^{-i\delta} G_1(\lambda) = \lambda^{2-\alpha} \left[\chi'' + \left(\frac{1}{2\lambda} - \frac{i}{4} \right) \chi' + \frac{i\alpha}{4\lambda} \chi \right] \tag{13.11}$$

We can satisfy the conditions within the fluid and on the free surface if we take as our function $G_1(\lambda)$ any single-valued function which is pure imaginary on the real axis and has singularities only on the real axis.

In order to determine the characteristic function of the appropriate wave motion, it is necessary to integrate differential equation (13.11) for the function $\chi(\lambda)$. In the most general case, wave motions possessing singularities on the free surface of the fluid are obtained.

If we assume that the fluid motion at the free boundary is regular for $t > 0$, then, since the combination

$$\frac{\partial^2 w}{\partial t^2} + ig \frac{\partial w}{\partial z}$$

is finite, the function $G_1(\lambda)$ reduces to an imaginary constant which we must put equal to zero by virtue of condition (13.2) and the additional condition that the pressure is independent of time as $y \to -\infty$.

Hence, we arrive at the problem of integrating the ordinary differential equation

$$\chi'' + \left(\frac{1}{2\lambda} - \frac{i}{4}\right)\chi' + \frac{i\alpha}{4\lambda}\chi = 0 \tag{13.12}$$

In this equation α is an arbitrary constant. It is not difficult to see that the solution of (13.12) also gives a certain wave motion for complex α. The fundamental solution, considered by N. E. Kochin, corresponds to the particular value $\alpha = -1/2$.

After the change of variable

$$\lambda = \frac{4}{i}\mu$$

equation (13.12) reduces to

$$\mu\frac{d^2\chi}{d\mu^2} + \left(\frac{1}{2} - \mu\right)\frac{d\chi}{d\mu} + \alpha\chi = 0 \tag{13.13}$$

The solution of (13.13) is expressed in terms of the confluent hypergeometric functions $y = M(k, \gamma, x)$ which satisfy the differential equation [32]:

$$xy'' + (\gamma - x)y' - ky = 0$$

The general solution of (13.13) is

$$\chi = C_1 M\left(-\alpha, \frac{1}{2}, \mu\right) + C_2\mu^{1/2}M\left(-\alpha + \frac{1}{2}, \frac{3}{2}, \mu\right)$$

Using this solution for the characteristic function of the wave motion, we can write

$$w = A_1w_1 + A_2w_2 \tag{13.14}$$

where

$$w_1(z, t, \alpha) = z^\alpha M\left(-\alpha, \frac{1}{2}, \frac{igt^2}{4z}\right) \tag{13.15}$$

$$w_2(z, t, \alpha) = z^\alpha\sqrt{\frac{igt^2}{4z}}\, M\left(-\alpha + \frac{1}{2}, \frac{3}{2}, \frac{igt^2}{4z}\right) \tag{13.16}$$

The arbitrary constants A_1 and A_2 may be complex.

Particular solutions of the wave problem (13.15) and (13.16) which depend on the single arbitrary constant α can be generalized by replacing t by $t - t_0$ and z by $z - z_0$. The constants t_0 and z_0 (z_0 is real) determine a change in the initial instant and the displacement of the singular point corresponding to the origin in the plane of flow. More general solutions can be constructed by superposition of these solutions: the constants A_1 and A_2 can be considered functions of the parameters α, t_0, and z_0 while the summation is being carried out.

From formulas (13.15) and (13.16), it is easy to deduce the following properties of the functions w_1 and w_2.

At $t = 0$, we have

$$w_1(z,\ 0,\ \alpha) = z^{\alpha},\quad \left(\frac{\partial w_1}{\partial t}\right)\bigg|_{t=0} = 0 \tag{13.17}$$

and

$$w_2(z,\ 0,\ \alpha) = 0,\quad \left(\frac{\partial w_2}{\partial t}\right)_{t=0} = \sqrt{\frac{ig}{4}}\cdot z^{\alpha-1/2} \tag{13.18}$$

Now, let us consider the wave motions for which the characteristic functions are determined by the formulas

$$\Omega_1(z,\ t) = -\frac{e^{\pi i\alpha}}{\pi i}\int_{-\infty}^{+\infty} f(x_0)\, w_1(z-x_0,\ t,\ \alpha)\, dx_0 \tag{13.19}$$

$$\Omega_2(z,\ t) = -\frac{2e^{\pi i}(\alpha-1/2)}{\sqrt{ig}\cdot\pi i}\int_{-\infty}^{+\infty} F(x_0)\, w_2(z-x_0,\ t,\ \alpha)\, dx_0 \tag{13.20}$$

where $f(x_0)$ and $F(x_0)$ are certain functions for which the integrals in formulas (13.19) and (13.20) converge.

At $t = 0$, we have

$$\Omega_1 = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(x_0)\, dx_0}{(x_0-z)^{-\alpha}},\qquad \frac{\partial\Omega_1}{\partial t} = 0 \tag{13.21}$$

and

$$\Omega_2 = 0,\qquad \frac{\partial\Omega_2}{\partial t} = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{F(x_0)\, dx_0}{(x_0-z)^{1/2-\alpha}} \tag{13.22}$$

The first of equations (13.21) yields for $\alpha = -1$ and $z = x$

$$\Omega_1(x) = \Phi_1 + i\Psi_1 = f(x) - \frac{1}{\pi i}\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{f(x_0)\, dx_0}{x_0 - x}$$

where the Cauchy principal value of the integral is taken. Hence, if $f(x)$ is real, then (13.19) gives the solution of the problem of the wave motion for the following initial conditions when $\alpha = -1$:

$$\Phi_1(x_0) = f(x),\qquad \frac{\partial\Phi_1}{\partial t} = 0 \quad \text{for} \quad t = 0$$

The following initial conditions are obtained for the function $\Psi(x, y)$:

$$\Psi_1(x,\ 0) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{f(x_0)\, dx_0}{x_0 - x},\qquad \frac{\partial\Psi_1}{\partial t} = 0 \ \text{ for } \ t = 0$$

If $f(x)$ is a pure imaginary function, then the initial conditions have analogous form but the roles of the functions Φ_1 and Ψ_1 are interchanged.

The second of equations (13.22) yields for $t = 0$ and $z = x$ when $\alpha = -1/2$:

$$\frac{\partial \Omega_2}{\partial t} = \frac{\partial (\Phi_2 + i\Psi_2)}{\partial t} = F(x) - \frac{1}{\pi i}\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{F(x_0)\,dx_0}{x_0 - x}$$

Hence, it follows that if $F(x)$ is real, the following initial conditions are obtained:

$$\Phi_2 + i\Psi_2 = 0, \qquad \frac{\partial \Phi_2}{\partial t} = F(x) \text{ for } t = 0 \text{ and } y = 0$$

This case was investigated by N. E. Kochin.

Now, let us clarify the character of the initial conditions in the general case when $\alpha \neq -1, -1/2$.

Let us consider the initial conditions for the function $\Omega_1(z, t)$. We have

$$\Omega_1(z, 0) = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} (x_0 - z)^{\alpha} f(x_0)\,dx_0$$

If $\alpha > 0$ and is an integer, then evidently $\Omega_1(z, 0)$ is a polynomial of degree α. In general, if $\alpha > 0$, then the function $\Omega_1(z, t)$ will become infinite as $z \to \infty$; consequently, the condition that the velocity vanish as $y \to -\infty$ is not satisfied for $\alpha \geqslant 1$.

If $\alpha = -(1 + s)$, where s is a positive integer, then we have

$$\Omega_1(z, 0) = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(x_0)\,dx_0}{(x_0 - z)^{1+s}} \tag{13.23}$$

Let us assume that the integral $\int_{-\infty}^{+\infty} f(x_0)\,dx_0$ is finite[1]) and introduce the following function:

$$\omega(z) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x_0)}{x_0 - z}\,dx_0 \tag{13.24}$$

[1]) It is evident that the expression for $\Omega_1(z)$ can have meaning in a number of cases when the integral in (13.24) diverges.

Under the assumptions accepted for the function $f(x_0)$, it is easy to confirm the validity of the relations

$$\frac{d^s\omega}{dz^s} = \omega^s(z) = \Gamma(s+1)\,\Omega_1(z, 0) \tag{13.25}$$

$$\omega(z) = \Gamma(s+1)\int_{-i\infty}^{z} dz_s \int_{-i\infty}^{z_s} dz_{s-1}\ldots\int_{-i\infty}^{z_2} \Omega(z_1)\,dz_1 = \int_{-i\infty}^{z} (z-u)^s\,\Omega_1'(u)\,du \tag{13.26}$$

Therefore, we have the following relation to determine $f(x)$:

$$\int_{-i\infty}^{x} (x-u)^s\,\Omega_1'(u)\,du = f(x) - \frac{\mathrm{PV}}{\pi i}\int_{-\infty}^{+\infty} \frac{f(x_0)\,dx_0}{x_0-x} \tag{13.27}$$

We have derived formula (13.27) from formulas (13.23) and (13.24) for integral $s>0$. This relation also remains valid for any real $s>-1$.

Indeed, from (13.23) we have

$$\Omega'(u) = -\frac{s+1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(x_0)\,dx_0}{(x_0-u)^{s+2}}$$

Multiplying by $(z-u)^s$ and integrating, we obtain

$$\int_{-i\infty}^{z} \Omega'(u)\,(z-u)^s\,du = -\frac{s+1}{\pi i}\int_{-\infty}^{+\infty} f(x_0)\,dx_0 \int_{-i\infty}^{z} \frac{(z-u)^s\,du}{(x_0-u)^{s+2}}$$

We make the following change of variables in the inner integral:

$$u = x_0 + (z-x_0)\,\frac{1}{\lambda}$$

whence

$$x_0 - u = \frac{x_0-z}{\lambda}, \qquad z-u = \frac{x_0-z}{\lambda}(1-\lambda), \qquad du = (x_0-z)\frac{d\lambda}{\lambda^2}$$

Substituting it into the inner integral, we obtain

$$\frac{1}{x_0-z}\int_0^1 (1-\lambda)^s\,d\lambda = \frac{1}{(s+1)(x_0-z)}$$

Using this, we find

$$\int_{-i\infty}^{z} (z-u)^s\,\Omega'(u)\,du = -\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(x_0)\,dx_0}{x_0-z}$$

Hence, formula (13.27) follows as $z\to x$.

Formula (13.25) can be generalized to fractional s if the fractional derivative $\omega^s(z)$ is defined by the formula

$$\omega^s(z) = \frac{1}{\Gamma(1-s)} \int_{-i\infty}^{z} (z-u)^{-s} \omega'(u)\, du$$

Indeed,

$$\omega^s(z) = -\frac{1}{\pi i \Gamma(1-s)} \int_{-\infty}^{+\infty} f(x_0)\, dx \int_{-i\infty}^{z} \frac{(z-u)^{-s}\, du}{(x_0-u)^2}$$

In order to evaluate the inner integral, let us make the change of variable

$$u = x_0 + (z - x_0)\frac{1}{\lambda}$$

after which we obtain

$$\int_{-i\infty}^{z} \frac{(z-u)^{-s}}{(x_0-u)^2} du = \frac{1}{(x_0-z)^{s+1}} \int_{0}^{1} \lambda^s (1-\lambda)^{-s}\, d\lambda = \frac{B(s+1, 1-s)}{(x_0-z)^{s+1}}$$

since

$$B(s+1, 1-s) = \frac{\Gamma(s+1)\Gamma(1-s)}{\Gamma(2)} \quad \text{and} \quad \Gamma(2) = 1$$

then

$$\omega^s(z) = -\frac{\Gamma(s+1)}{\pi i} \int_{-\infty}^{+\infty} \frac{f(x_0)\, dx_0}{(x_0-z)^{s+1}} = \Gamma(s+1)\,\Omega(z)$$

§ 14. Three-Dimensional Self-Similar Motions of Continuous Media

The formulation of the problems of motion of an incompressible fluid in §§ 12 and 13, which led to a reduction in the number of independent variables, can be extended and generalized.

The motion of a compressible medium in which the dimensionless parameters depend only on the combinations

$$\frac{x}{bt^\delta}, \quad \frac{y}{bt^\delta}, \quad \frac{z}{bt^\delta}$$

(where x, y, and z denote the Cartesian coordinates, t is the time, and b is a constant with dimensions $LT^{-\delta}$) will be called self-similar with the centre of similarity at the origin of the coordinate system.

It is easy to discover the general character of all problems for which self-similarity exists: it is sufficient for a system of dimensional characteristic parameters prescribed by supplementary conditions, and among them by boundary or initial conditions, to con-

tain not more than two constants with independent dimensions other than length or time.

In other words, the system of characteristic parameters should be represented by a table

$$a,\ b,\ x,\ y,\ z,\ t,\ \alpha_1,\ \alpha_2,\ \ldots$$

where α_1, α_2, ... are arbitrary combinations of dimensional constants. There can be any number of these, and the constants a and b have dimensions of the form $[a] = \mathrm{ML}^k\mathrm{T}^s$, $[b] = \mathrm{LT}^{-\delta}$. Here $\delta \neq 0$, and k and s can be arbitrary. Without loss of generality, the constant a can always be replaced by $A = ab^{s/\delta}$ with the dimensions $[A] = \mathrm{ML}^{\omega-3}$, where ω can be arbitrary.

Generally speaking, for self-similarity to exist in the motion of a continuous medium it is necessary that the formulation of the problem should not contain a characteristic length or time (wedge, cone, etc.).

We shall now give some examples of self-similar motion.

(1) The problem of the outward motion of an infinite mass of an incompressible fluid, initially at rest, by a cavity which expands from a point under conditions preserving geometrical similarity.

Suppose that the prescribed radial velocities of the interior boundary of the fluid are determined by the equation

$$v(t,\ \theta,\ \psi) = bf(\theta,\ \psi)\,t^{\delta-1}$$

where θ and ψ are polar coordinates, and b is a constant with dimensions: $[b] = \mathrm{LT}^{-\delta}$. For $\delta > 0$ the interior surface extends continuously and under conditions of similarity from the origin. The perturbed motion of the incompressible fluid is potential and is determined by the following system of parameters:

$$\rho,\ b,\ r,\ \theta,\ \psi,\ t,\ f(\theta,\ \psi)$$

where ρ is the density, and r is the polar radius; the initial pressure and the pressure at infinity p_0 are indeterminate so that only pressure differences $p - p_0$ can be considered.

It is evident that in this case the velocity potential is of the form

$$\varphi = rbt^{\delta-1}\Phi\left(\theta,\ \psi,\ \frac{r}{bt^{\delta}}\right)$$

The problem can be formulated in a different way; instead of prescribing the expansion velocity of the cavity we can give the pressure acting at the surface of this cavity:

$$p - p_0 = \rho b^2 t^{2(\delta-1)} F(\theta,\ \psi)$$

One such problem will be considered in § 11, Chapter IV.

(2) The above considerations concerning the self-similarity of a fluid carry over to the case when the motion of the fluid with a free surface is considered, on condition that this surface is initially a plane and that the centre of symmetry for the motion of the expanded prescribed surface lies in that plane. In this way, we can generalize problems on the penetration by a cone or a wedge, in the case of arbitrary but not rigid surfaces, which change their shape under the conditions of similarity. We can also treat bodies displacing a fluid bounded by a free surface.

In particular, it is possible to consider self-similarity conditions when a wedge or a cone enters into a fluid at a velocity varying with time as a power function.

If the weight of a fluid is taken into account, it is necessary to include, among the characteristic parameters, the acceleration due to gravity. To preserve self-similarity it is necessary to have $[g] = [b]$, i.e. $\delta = +2$.

Consequently, for a wedge or a cone entering with uniform acceleration into an incompressible heavy fluid, the perturbed motion of the fluid will be self-similar.

Conditions of the self-similarity in a fluid with a free surface are also preserved in cases when at the initial instant the free surface is conic or wedge-shaped and when the centre of similarity coincides with the vertex of the cone or the wedge.

As an example of a self-similar solution for an elastic medium we should like to point to Boussinesq's problem on the distribution of stress and strain in the elastic half-space bounded by a plane near the point of application of a prescribed concentrated force P.

In problems of statics, the properties of an elastic medium are completely determined by Young's modulus E and Poisson's ratio σ. The external force is prescribed by the magnitude of P and by the abstract parameters determining its direction.

We take the polar coordinates r and θ with the origin at the point of application of the concentrated load in a plane which contains the force vector and which is perpendicular to the boundary. The system of characteristic parameters is:

$$P,\ E,\ r,\ \theta,\ \psi,\ \sigma,\ \theta_0$$

θ_0 is the angle defining the direction of the force P. Owing to the linear character of the problem all stresses and strains are linear functions of P and hence the dependence on P is known beforehand; the dependence of all quantities on E and r can be determined immediately from dimensional considerations: this leads to only two independent variables θ and ψ. In the case of axial symmetry (P is perpendicular to the boundary plane), the variable ψ will disappear and therefore it is easy to obtain a complete solution of the problem for which the integration of one differential equation is required.

In some cases, which are obvious whenever a particular example is considered, the above arguments can be extended to include various problems of unsteady motion of compressible media.

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CHAPTER III

APPLICATIONS TO THE THEORY OF MOTION OF A VISCOUS FLUID AND TO THE THEORY OF TURBULENCE

§ 1. Diffusion of Vorticity in a Viscous Fluid

In this and succeeding sections we shall give examples of the use of dimensional analysis in the mathematical solution of some physical problems.

We begin with the problem of diffusion of vorticity in the one-dimensional unsteady motion of a viscous incompressible fluid of infinite extent [1]. We suppose that at the initial instant $t = 0$ potential motion exists everywhere except at the pole O which is the trace on the plane of the motion of an infinite rectilinear point vortex with the associated circulation Γ.

We assume that the motion has axial symmetry and denote the angular velocity of the fluid by Ω.

As is known, the circulation along a circle of radius R centred at O is

$$\Gamma_R = 2\int_0^R \int_0^{2\pi} \Omega r\, dr\, d\theta = 4\pi \int_0^R r\Omega(r)\, dr \tag{1.1}$$

At the initial instant, for any circle, no matter how small, we have

$$\Gamma_R = \Gamma \tag{1.2}$$

In the case under consideration, the equation governing the propagation of vorticity will be

$$\frac{\partial \Omega}{\partial t} = \nu \left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega}{\partial r} \right) \tag{1.3}$$

where ν is the coefficient of kinematic viscosity ($\nu = \mu/\rho$). The problem is to determine Ω as a function of the radius r and the time t.

It follows from the formulation of the problem that

$$\Omega = f(\Gamma, \nu, r, t)$$

Since equation (1.3) is linear, the initial condition gives that Ω is proportional to Γ, that is,

$$\Omega = \Gamma f_1(\nu, r, t) \tag{1.4}$$

The dimensionless combination $\Omega\nu t/\Gamma$ must be expressed as a function of the single independent dimensionless quantity $r^2/(\nu t) = \xi$, which must be composed of the dimensional parameters ν, r, and t. Therefore,

$$\Omega = \frac{\Gamma}{\nu t}\,\psi(\xi) \tag{1.5}$$

It is evident from (1.5) that partial differential equation (1.3) for Ω, involving two independent variables r and t, reduces to an ordinary differential equation in the one unknown ξ.

Substituting the expression for Ω from (1.5) into (1.3), we have:

$$\psi(\xi) + \xi\psi'(\xi) + 4[\psi'(\xi) + \xi\psi''(\xi)] = 0$$

Integrating, we obtain

$$\xi\psi + 4\xi\psi' = C$$

The constant C is zero for the solution in which $\psi(0)$ and $\psi'(0)$ are finite. Integrating the equation

$$4\frac{d\psi}{d\xi} + \psi = 0$$

we find

$$\psi = A\exp\left(-\frac{\xi}{4}\right)$$

This yields for the magnitude of the vorticity Ω:

$$\Omega = \frac{\Gamma}{\nu t} A\exp\left(-\frac{r^2}{4\nu t}\right)$$

We determine the constant A from the initial conditions. The circulation along a circle of radius R is

$$\Gamma_R = 4\pi\frac{A\Gamma}{\nu t}\int_0^R r\exp\left(-\frac{r^2}{4\nu t}\right)dr = 8\pi A\Gamma\left[1-\exp\left(-\frac{R^2}{4\nu t}\right)\right] \tag{1.6}$$

For $t = 0$ and any $R > 0$, we have

$$\Gamma_R = 8\pi A\Gamma$$

The initial condition $\Gamma_R = \Gamma$ yields

$$A = \frac{1}{8\pi}$$

The final solution of the problem is

$$\Omega = \frac{\Gamma}{8\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right) \tag{1.7}$$

Let us denote the fluid velocity by $\mathfrak{v}(r, t)$. The motion has axial symmetry, hence, the fluid velocity is directed normal to a radius-vector drawn from the pole O to the point under consideration.

Taking into account the direction of the velocity vector, we obtain the following relation between Γ_R and $\mathfrak{v}$:

$$\Gamma_R = 2\pi r \mathfrak{v}$$

Using (1.6), we find that the velocity distribution over the radius r for the time t is

$$\mathfrak{v} = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right]$$

A velocity distribution corresponding to a point vortex in an ideal fluid is obtained for $t = 0$. The fluid motion is potential for $r > 0$ and $t = 0$ and vortices are absent; the fluid motion is vortex at each point of the fluid for $r > 0$ and $t > 0$. Formula (1.7) yields the law of the propagation or diffusion of vorticity: this formula shows that the strength of the vortex at each point increases with time from zero to a maximum equal to $\Gamma/(2\pi r^2 e)$ and then again tends to zero.

Since (1.3) is linear, then, starting from the solution obtained for the propagation of a point vortex, a solution of the problem of symmetric motion for any initial velocity distribution can be constructed by the principle of superposition.

§ 2. Exact Solutions of the Equations of Motion of a Viscous Incompressible Fluid

We consider the steady motion of a viscous incompressible fluid of infinite extent.

The Navier-Stokes equation and the continuity equation can be written in the form

$$\left.\begin{aligned} \bar{\mathfrak{v}} \times \nabla \mathfrak{v} &= -\operatorname{grad}\left(\frac{p}{\rho} - U\right) + \nu \nabla \bar{\mathfrak{v}} \\ \operatorname{div} \bar{\mathfrak{v}} &= 0 \end{aligned}\right\} \qquad (2.1)$$

In what follows we use spherical polar coordinates. The independent variables and the characteristic parameters will be

$$r, \ \theta, \ \lambda, \ \nu$$

where r is the distance of the point under consideration to the pole, θ is the altitude, λ is the longitude, and ν is the coefficient of kinematic viscosity. The quantities to be determined will be the velocity components v_r, v_θ, and v_λ and the dynamic pressure referred to the density and equal to $(p/\rho) - U$.

Let us study the solutions of (2.1) which are completely determined by the parameters r, θ, λ, and ν plus a single dimensional constant A.

Let the dimensions of A be of the form

$$[A] = \mathrm{L}^p\mathrm{T}^q$$

where p and q are constants.

Under this assumption, it is evident that all the dimensionless combinations of the quantities introduced will be functions of only the three abstract parameters:

$$\theta,\ \lambda,\ \pi = \frac{r^{p+2q}\nu^{-q}}{A}$$

Then the desired functions can be represented as

$$v_r = \frac{\nu}{r} f(\pi, \lambda, \theta), \qquad v_\theta = \frac{\nu}{r} \varphi(\pi, \lambda, \theta)$$

$$v_\lambda = \frac{\nu}{r} \psi(\pi, \lambda, \theta), \quad U - \frac{p}{\rho} = \frac{\nu^2}{r^2} F(\pi, \lambda, \theta)$$

The most general solution of (2.1) can be represented and considered in such a form.

If we assume that

$$p + 2q = 0$$

the number of independent variables will be reduced.

This condition means that the dimensions of the constant A are a certain power of the dimensions of the kinematic viscosity coefficient ν.

In addition to the above assumption, let us assume that the motions to be studied have axial symmetry so that the variable λ is not essential.

The following formulas result from these assumptions:

$$v_r = \frac{\nu}{r} f(\theta), \quad v_\theta = \frac{\nu}{r} \varphi(\theta), \quad v_\lambda = \frac{\nu}{r} \psi(\theta), \quad U - \frac{p}{\rho} = \frac{\nu^2}{r^2} F(\theta) \qquad (2.2)$$

These formulas give the velocity field and pressure as functions of the variable r. In this case, from (2.1) a system of ordinary nonlinear differential equations is obtained for the four functions f, φ, ψ, and F:

$$\left.\begin{aligned} f'' + f'(\cot\theta - \varphi) + f^2 + \varphi^2 + \psi^2 - 2F = 0 \\ \varphi\varphi' - \psi^2\cot\theta - f' - F' = 0 \\ \psi'' - \varphi\psi' - \varphi\psi\cot\theta + \psi'\cot\theta - \frac{\psi}{\sin^2\theta} = 0 \\ f + \varphi' + \varphi\cot\theta = 0 \end{aligned}\right\} \qquad (2.3)$$

After elimination of F and after certain simple transformations, we obtain:

$$\left.\begin{aligned} & f''' + 2\psi(\psi' + \psi\cot\theta) + (f'\cot\theta)' - (\varphi f')' + 2ff' + 2f' = 0 \\ & \varphi(\psi' + \psi\cot\theta) = (\psi' + \psi\cot\theta)' \\ & \qquad\qquad f = -(\varphi' + \varphi\cot\theta) \end{aligned}\right\} \tag{2.4}$$

The general solution of this system of equations depends on six arbitrary constants.

Before proceeding to the study of the solution of system of equations (2.4) we note certain general properties of the viscous fluid motion under consideration.

The differential equations for the projections of the streamlines on the meridian plane can be written as

$$\frac{dr}{v_r} = \frac{r\,d\theta}{v_\theta} \quad \text{or} \quad \frac{dr}{f(\theta)} = \frac{r\,d\theta}{\varphi(\theta)}$$

whence

$$\ln\frac{r}{a} = \int \frac{f(\theta)\,d\theta}{\varphi(\theta)}$$

where a is a constant of integration. From the last equation of system (2.4), we obtain

$$\frac{r}{a} = \frac{1}{\varphi\sin\theta} \tag{2.5}$$

From general considerations of dimensional analysis and also directly from (2.5), it follows that the streamlines of the flow are similar curves.

Let us denote the mass flow rate and the momentum flux through the closed surface S by Q and $\mathbf{J}$, respectively, i.e.

$$Q = \int_S \mathfrak{v}_n\,d\sigma, \qquad \mathbf{J} = \int_S \bar{\mathfrak{v}}\mathfrak{v}_n\,d\sigma$$

The dimensions of Q and J are given by the formulas

$$[Q] = \mathrm{L}^3\mathrm{T}^{-1}, \quad [J] = \mathrm{L}^4\mathrm{T}^{-2}$$

When the surface S contracts to a point at the pole, we find that Q and $\mathbf{J}$ can only depend on the coefficient ν and on the constant A the dimensions of which can be expressed in terms of the dimensions of ν. Since the dimensions of Q and ν are independent, it is evident that the mass flow rate Q equals either zero or infinity. The dimensions of J are expressed in terms of the dimensions of the coefficient ν; consequently, J can be finite.

Another situation holds in two-dimensional motions. If the velocity field over the whole plane depends only on the point coordinates and on a constant with dimensions dependent on the dimensions

of the kinematic viscosity coefficient ν for plane motion, then formulas analogous to (2.2) will be valid in polar coordinates. (In this case, r is a radius-vector in the plane of motion.)

The mass flow rate and the momentum flux for plane motion can be determined by

$$Q=\int\limits_L \mathfrak{v}_n\,dS, \qquad \mathbf{J}=\int\limits_L \bar{\mathfrak{v}}\mathfrak{v}_n\,dS$$

where L is a certain closed contour enclosing the origin. In this case, the dimensions of Q and J are represented by

$$[Q]=\mathrm{L^2T^{-1}}, \qquad [J]=\mathrm{L^3T^{-1}}$$

Therefore, plane motions of the kind considered can be characterized by a finite mass flow rate but the corresponding momentum will be equal to zero or infinity.

This result enabled Hamel and a number of other authors to obtain exact solutions of the Navier-Stokes equations in the problem of the fluid motion in an angle between two planes by reducing them to ordinary differential equations [2].

We now consider the solution of system of equations (2.4). The first of equations (2.4) can be written

$$\left[\frac{1}{\sin\theta}\left(\frac{\Phi'}{\sin\theta}\right)'\right]=\frac{(\psi^2\sin^2\theta)'}{\sin^2\theta} \tag{2.6}$$

where

$$\Phi=\left(\varphi'-\frac{1}{2}\varphi^2\right)\sin^2\theta-\varphi\sin\theta\cos\theta \tag{2.7}$$

The following integral can be derived from equation (2.6):

$$\psi^2\sin^2\theta-\sin\theta\left(\frac{\Phi'}{\sin\theta}\right)'+2\Phi+2\cot\theta\cdot\Phi'=D \tag{2.8}$$

where D is a constant of integration.

The function $\psi(\theta)$ determines the distribution of the velocity components perpendicular to the meridian plane.

It is not difficult to see that system of equations (2.4) yields the same relations to determine the functions f and φ both for $\psi=0$ or for $\psi\sin\theta=\text{const}$.

The condition $\psi=c/\sin\theta$ yields $v_\lambda=c\nu/(r\sin\theta)$; the velocity field for v_λ corresponds to a vortex line coincident with the axis of symmetry. Therefore, the equations of motion will be satisfied if we add the velocity field of a vortex line to any velocity field of the kind considered.

If we assume that

$$\psi\sin\theta=\text{const} \tag{2.9}$$

then equation (2.6) can be integrated three times, and the solution of the problem reduces to the integration of Riccati's equation

$$\left(\varphi' - \frac{1}{2}\varphi^2\right)\sin^2\theta - \varphi\sin\theta\cos\theta = M\cos 2\theta + N\cos\theta + R \qquad (2.10)$$

where M, N, and R are arbitrary constants of integration [3].

If we put $M = N = R = 0$, then (2.10) is integrated easily and yields

$$\varphi = \frac{2\sin\theta}{A+\cos\theta}, \quad \text{whence } f = -2 + \frac{2(A^2-1)}{(A+\cos\theta)^2} \qquad (2.11)$$

where A is an arbitrary constant of integration. Solution (2.11) was obtained by Landau [4].

It is not difficult to confirm that the mass flow rate through a surface enclosing the origin equals zero for this solution when $|A| > 1$ and is infinite when $|A| < 1$.

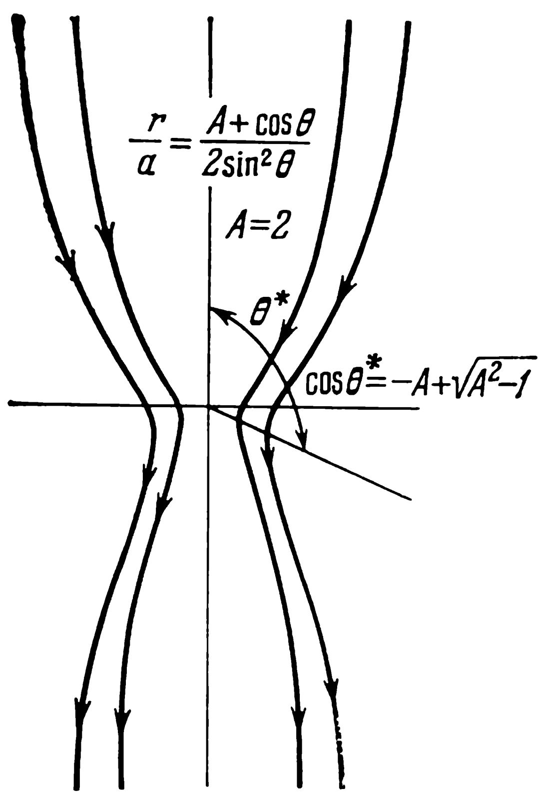

Fig. 20. Streamlines for a source of zero intensity and finite momentum in a viscous fluid.

The flow corresponding to the component of momentum along the axis of symmetry through a sphere with centre at the origin is given by

$$J = \rho\nu^2\left[8(A^2-1)\ln\frac{A-1}{A+1} + 8A - \frac{32A}{3(A^2-1)} + \frac{8A(3A^2-1)}{A^2-1}\right] \qquad (2.12)$$

Therefore, the magnitude of the momentum J is independent of the radius of the sphere and contains the mechanical characteristics of a singular point at the origin. The equations of the streamlines in this flow have the form

$$\frac{r}{a} = \frac{A+\cos\theta}{2\sin^2\theta}$$

The shape of the streamline for $A > 1$ is shown in Fig. 20.

At infinity, the streamlines approach parabolic form. The radius-vector r, for motion along a streamline, attains a minimum value at a certain $\theta = \theta^*$ defined by the relation:

$$\cos\theta^* = -A + \sqrt{A^2-1}$$

This corresponds to the motion of an infinite viscous fluid caused by a fine jet at the origin which is flowing from the end of an infinitely thin pipe with finite momentum in the direction of the x-axis.

A whole series of solutions of Riccati's equation (2.10) can be obtained for particular values of the constants M, N, and R. For exam-

ple, for $R = 1$, $N = 0$, and $M = 1/2$, we have the solution

$$\varphi = -\left(\cot\theta + \coth\frac{\theta+\theta_0}{2}\right)$$

where θ_0 is an arbitrary constant. For

$$N = 0, \quad M = \frac{1}{4} - m^2, \quad \text{and} \quad R = \frac{9}{4} - (m+1)^2$$

we have the solution

$$\varphi = (2m-1)\cot\theta - \frac{2\sin^{2m}\theta}{\int \sin^{2m}\theta\, d\theta}$$

Riccati's equation (2.10) can be solved in the general case by using hypergeometric functions [5].

By changing for the variables

$$\varphi = -2\frac{y'(\theta)}{y(\theta)}, \quad \mu = \cos^2\frac{\theta}{2}$$

equation (2.10) is reduced to

$$\frac{d^2y}{d\mu^2} + \frac{\frac{M+R-N}{2} + (N-4M)\mu + 4\mu^2 M}{4\mu^2(\mu-1)^2}\, y = 0 \qquad (2.13)$$

This equation is easily integrated in terms of hypergeometric functions; its general integral can be written as:

$$Y(\theta) = \left(\cos\frac{\theta}{2}\right)^{\gamma}\left(\sin\frac{\theta}{2}\right)^{1+\alpha+\beta-\gamma}\left\{PF\left(\alpha, \beta, \gamma, \cos^2\frac{\theta}{2}\right)\right.$$
$$\left. + Q\left(\cos^2\frac{\theta}{2}\right)^{1-\gamma} F\left(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma, \cos^2\frac{\theta}{2}\right)\right\} \qquad (2.14)$$

where the constants α, β, and γ are related to M, N, and R by the formulas:

$$M = \frac{1-(\alpha-\beta)^2}{4}$$
$$N = 1-(\alpha+\beta)^2 + 2\gamma(\alpha+\beta-1)$$
$$R = \frac{3[1-(\alpha+\beta)^2]}{4} - \alpha\beta - 2\gamma^2 + 2\gamma(\alpha+\beta+1)$$

The parameters α, β, and γ can be used as arbitrary constants instead of M, N, and R. (If γ is an integer, then the solution can also be written in another way. To do this, the representation of the solution of the hypergeometric equation in the form cited in [6] can be used.)

The solution obtained for φ depends on four arbitrary constants, on the three parameters α, β, and γ, and on the ratio P/Q.

Equation (2.13) degenerates into the equation $y''(\mu) = 0$ for $M = N = R = 0$, and then has only one regular singular point at $\mu = \infty$. The corresponding solution was considered above.

If $N=-4M=-4R/3$, then the factor $(\mu-1)^2$ in the denominator of (2.13) cancels out and only two regular singular points $\mu = 0$ and $\mu = \infty$ remain. In this case, (2.13) transforms into the Euler equation

$$\mu^2 \frac{d^2y}{d\mu^2}+My=0$$

which is integrated easily. (In analogy to the discussed motions of a viscous fluid, meridional flows of an incompressible conducting fluid also belong to the class of self-similar motions defined by a single dimensional constant $[A] = \mathrm{L}^p\mathrm{T}^q\mathrm{M}$ for $p+2q+3=0$ [7].)

§ 3. Boundary Layer in the Flow of a Viscous Fluid Past a Flat Plate

We consider the flow of an incompressible viscous fluid past an infinitely thin plate. We suppose that the fluid moves forward at the constant velocity U_0 far in front of the plate; the plate is of infinite length and is parallel to the unperturbed velocity direction. The problem is one of plane steady motion; the fluid occupies the whole plane beyond the plate. It is one of the simplest problems concerning the motion of a viscous fluid; nevertheless, it has not been solved as an exact solution of the Navier-Stokes equations because of considerable mathematical difficulties. We analyse this problem by using the Prandtl equations, which are obtained from the general equations of the motion of a viscous fluid by using certain approximations [8].

The Prandtl boundary layer equations in the case under consideration are:

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=\nu\frac{\partial^2 u}{\partial y^2}\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&=0\end{aligned}\right\}\qquad(3.1)$$

where u and v are the components of the fluid velocity along the coordinate axes, and ν is the kinematic viscosity coefficient. The x-axis is directed along the plate, and the y-axis is perpendicular to the plate. In addition to (3.1), we have the boundary conditions:

$$\left.\begin{aligned} &u=v=0 \quad &&\text{for}\quad x>0,\ y=0\\ &u=U_0 \quad &&\text{for}\quad y=\pm\infty\end{aligned}\right\}\qquad(3.2)$$

to determine $u(x, y)$ and $v(x, y)$.

The characteristic parameters will be

$$U_0,\ \nu,\ x,\ y$$

Since the plate is flat and infinitely long, it is impossible to choose a characteristic linear dimension. It follows from the general con-

siderations of dimensional analysis that all the dimensionless quantities are functions of the two dimensionless combinations:

$$\frac{y}{x}, \quad \frac{y}{\sqrt{\nu x/U_0}}$$

Consequently, formulas of the type

$$u = U_0 f\left(\frac{y}{x}, \quad \frac{y}{\sqrt{\nu x/U_0}}\right) \tag{3.3}$$

$$v = \sqrt{\frac{\nu U}{x}}\,\Phi\left(\frac{y}{x}, \frac{y}{\sqrt{\nu x/U_0}}\right) \tag{3.4}$$

apply in this problem.

We now show that the first parameter y/x in (3.3) and (3.4) is not essential because of the singularities of (3.1). To do this we make the transformation of the variables:

$$x = l\xi, \quad y = \sqrt{\frac{\nu l}{U_0}}\,\eta, \quad u = U_0 u_1, \quad v = \sqrt{\frac{\nu U_0}{l}}\,v_1 \tag{3.5}$$

where l is a constant larger than zero. If we attribute the dimensions of length to l, then the quantities ξ, η, u_1, and v_1 can be considered dimensionless.

When the new variables defined by (3.5) are introduced, equations (3.1) become

$$\left.\begin{aligned} &u_1\frac{\partial u_1}{\partial \xi} + v_1\frac{\partial u_1}{\partial \eta} = \frac{\partial^2 u_1}{\partial \eta^2} \\ &\frac{\partial u_1}{\partial \xi} + \frac{\partial v_1}{\partial \eta} = 0 \end{aligned}\right\} \tag{3.6}$$

Boundary conditions (3.2) become, in terms of the new variables:

$$\left.\begin{aligned} &u_1 = v_1 = 0 \quad &&\text{for} \quad \xi > 0,\ \eta = 0 \\ &u_1 = 1 \quad &&\text{for} \quad \eta = \pm\infty \end{aligned}\right\} \tag{3.7}$$

Equations (3.6) and boundary conditions (3.7) give the formulation of the boundary layer problem in dimensionless form. The solution of this problem cannot depend on $U_0 l/\nu = \mathrm{Re}$ which appears neither in equations (3.6) nor in boundary conditions (3.7). (This is a property of the Prandtl equations (3.1). If transformation (3.5) is applied to the Navier-Stokes equations, then we obtain dimensionless equations containing the parameter Re; consequently, subsequent conclusions lose their validity when applied to the Navier-Stokes equations.) On the other hand, general formulas (3.3) and (3.4) show that

$$u_1 = \frac{u}{U_0} = f\left(\frac{\eta}{\xi\sqrt{\mathrm{Re}}}, \quad \frac{\eta}{\sqrt{\xi}}\right) \tag{3.8}$$

$$v_1\sqrt{\xi} = \frac{v}{\sqrt{\nu U_0/x}} = \Phi\left(\frac{\eta}{\xi\sqrt{\mathrm{Re}}}, \quad \frac{\eta}{\sqrt{\xi}}\right) \tag{3.9}$$

Since the solution need not depend on Re, it follows that the first argument y/x cannot enter into the right-hand side of formulas (3.8) and (3.9).

Hence, we have proved that the solution of the formulated problem must have the form:

$$u = U_0 f\left(\frac{y}{\sqrt{\nu x/U_0}}\right) \tag{3.10}$$

$$v = \sqrt{\frac{\nu U_0}{x}}\,\Phi\left(\frac{y}{\sqrt{\nu x/U_0}}\right) \tag{3.11}$$

(The proof of the validity of formula (3.10) is made by another method in [9].) Now we introduce the new variable

$$\lambda = \frac{\eta}{\sqrt{\xi}} = \frac{y}{\sqrt{\nu x/U_0}}$$

and put $f(\lambda) = \varphi'(\lambda)$. Substituting u and v in the continuity equation, we express $\Phi(\lambda)$ in terms of $\varphi(\lambda)$ as follows:

$$\Phi'(\lambda) = \frac{1}{2}\lambda\varphi''(\lambda) = \frac{1}{2}(\lambda\varphi' - \varphi)'$$

Using this relation, formulas (3.10) and (3.11) can now be written in the form:

$$u = U_0\varphi'(\lambda) \tag{3.10$'$}$$

$$v = \sqrt{\frac{\nu U_0}{x}}\cdot\frac{1}{2}[\lambda\varphi'(\lambda) - \varphi(\lambda)] \tag{3.11$'$}$$

Substituting the expressions obtained for u and v in the first of equations (3.1), we obtain an ordinary third-order differential equation for $\varphi(\lambda)$:

$$2\varphi''' + \varphi\varphi'' = 0 \tag{3.12}$$

From boundary conditions (3.2) for the desired function $\varphi(\lambda)$ which satisfies (3.12), the following boundary conditions are obtained:

$$\varphi'(0) = \varphi(0) = 0 \quad \text{and} \quad \varphi'(\infty) = 1 \tag{3.13}$$

The solution of nonlinear differential equation (3.12) under boundary conditions (3.13) can be obtained approximately [10]. A very general property of (3.12) used by Töpfer in an approximate method of solution [11] is the following.

If $\varphi_0(\lambda)$ is a solution of (3.12), then the function

$$\varphi(\lambda) = a\varphi_0(a\lambda)$$

where a is an arbitrary constant, is also a solution of (3.12). It is easy to see by a direct check that this property is correct.

We take as the initial solution $\varphi_0(\lambda)$ that solution of (3.12) which satisfies the boundary conditions

$$\varphi_0(0) = \varphi_0'(0) = 0 \quad \text{and} \quad \varphi_0''(0) = 1$$

The function $\varphi_0(\lambda)$ can be constructed by the usual approximate methods. The following limit can be calculated from the approximate solution:

$$\lim_{\lambda \to +\infty} \varphi_0'(\lambda) = k$$

Numerical calculations yield $k = 2.0854$. The solution of (3.12) given by the formula

$$\varphi(\lambda) = \alpha^{1/3} \varphi_0(\alpha^{1/3}\lambda)$$

satisfies the boundary conditions

$$\varphi(0) = \varphi'(0) = 0 \quad \text{and} \quad \varphi''(0) = \alpha$$

in which

$$\lim_{\lambda \to +\infty} \varphi'(\lambda) = k \cdot \alpha^{2/3}$$

Hence, it is clear that it is sufficient to put

$$\alpha = \frac{1}{k^{3/2}} = 0.332$$

to obtain the desired solution.

Defining $\alpha = \varphi''(0)$, we easily find the friction drag acting on the plate by using (3.10′).

We have for the friction stress τ at the plate:

$$\tau = \mu \left(\frac{du}{dy} \right)_{y=0} = \mu U_0 \frac{\varphi''(0)}{\sqrt{\nu x / U_0}} = 0.332 \sqrt{\frac{\rho \mu U_0^3}{x}} \tag{3.14}$$

Using this, we calculate the drag W of a section of a plate of width b and of length l:

$$W = b \int_0^l \tau \, dx = 0.664 b \sqrt{\rho \mu l U_0^3} \tag{3.15}$$

The friction stress τ and the drag W are seen to be proportional to the three-halves power of the stream velocity.

The following formula for the friction coefficient c_f is obtained from relation (3.15):

$$c_f = \frac{2W}{\rho b l U_0^2} = \frac{1.328}{\sqrt{\mathrm{Re}}}$$

where

$$\mathrm{Re} = \frac{U_0 l}{\nu}$$

Experimental data [12] on plane smooth plates are in good agreement with the velocity and drag distributions in the laminar flow mode characterized by low values of the Reynolds number

$$\mathrm{Re} = \frac{U_0 l}{\nu} < 3 \times 10^5$$

The laminar steady motion considered above is unstable at high Reynolds numbers. Turbulent motion arises which basically alters the drag and velocity distributions near the plate.

§ 4. Isotropic Turbulent Motion of an Incompressible Fluid

1. Averaging of Turbulent Motion. Many fluid motions observed in nature and the majority of motions with which we deal in engineering are characterized by the presence of disordered unsteady fluid motions superposed on a basic fluid motion which can be represented as a certain statistically average motion. Fluid motions of such a kind are called turbulent.

The velocity, pressure, and other quantities at each point of the flow in turbulent fluid motion undergo irregular fluctuating variations about certain average values. Consequently, it is feasible to use probability theory concepts to investigate turbulent flows; in this case, the instantaneous values of the mechanical characteristics are considered random quantities, and the mean values are defined as the mathematical expectations [13]. Often, however, the mean values are determined as the usual time averages. The time interval over which the averaging is carried out must be sufficiently large in comparison with the typical time of an individual fluctuation and must be small in comparison with the time for a noticeable change to take place in the mean quantities if the averaged motion is not stationary [1].

The mean values of the pressure, the velocity components, the products of the fluctuating velocity components, evaluated at the same point or at neighbouring points (the correlation coefficient of the velocity), etc. depend to a great extent on the presence of turbulent mixing. This causes the variations in the mean quantities to be equalized and smoothed out to a degree which varies with position in space.

Experiment shows that steady laminar motions of a fluid at large values of the Reynolds number, i.e. at high speeds and large scales, become unstable and change to unsteady turbulent motions. In a number of cases, the mean characteristics of such motions are steady.

The investigation of turbulent flow is very important in modern hydro- and aerodynamics and affects many basic problems including the calculation of profile drag and flow in ducts.

In all theoretical investigations of viscous fluid motion, it is assumed that the Navier-Stokes equations apply to the actual unsteady pulsating motion. However, when turbulence is present, individual fluid particles execute very tortuous and complex motions; to obtain the solution of the Navier-Stokes equations would be an extremely awkward and complicated problem comparable to that of describing

the motion of the individual molecules of a large volume of gas. Consequently, the fundamental problems of turbulent fluid motion in hydromechanics are posed, exactly as in kinetic gas theory, as problems of finding out functional relations between mean quantities.

The equations of motion for the mean quantities can be obtained by averaging the equations for the quantities describing the instantaneous state of motion. Because these equations are nonlinear, after the averaging we are left with more unknowns than equations. The mean values of nonlinear terms, for example, the product of two or more quantities, are new unknowns [14]. Hence, when averaging the Navier-Stokes equations for an incompressible fluid (and in what follows we limit the analysis only to incompressible fluids), the mean values of the products $\overline{u_i u_k}$ $(i, k = 1, 2, 3)$ must be taken into account in addition to the mean values of the velocity components $\bar{u}_1$, $\bar{u}_2$, and $\bar{u}_3$ (following the custom, we shall denote the mean values by a bar).

Therefore, those vector momentum equations which are sufficient for studying real motions are inadequate for a mathematical study of an averaged turbulent motion. Consequently, a complete theoretical investigation of averaged turbulent motions is possible only if we make certain additional hypotheses the validity of which must be established by experiment. (We are discussing the formulation of mathematical problems with a finite number of unknowns.)

The essence of a number of papers on turbulence reduces to the study of the validity of various plausible hypotheses. These simple and natural hypotheses can be verified experimentally, and permit fundamental problems of turbulent fluid motion to be formulated and solved theoretically.

No general mathematical formulation of the problem of arbitrary averaged turbulent motions yet exists and, in general, the formulations of mathematical rheological problems dealing with averaged turbulent motions of liquids and gases are quite different, adjusted to individual, fairly limited classes of problems.

Dimensional analysis and similarity concepts are frequently used as the fundamental means of investigating turbulent fluid motions.

2. Properties of Homogeneity and Isotropy. Let us consider the turbulent motion of a viscous fluid of infinite extent (the following analysis also refers to fluid motion in a finite region if the effect of boundaries can be neglected).

The state of motion at each instant t is determined by the initial perturbations (the perturbations in the fluid at $t = 0$) and by the inertial and viscous properties of the fluid, i.e. by the quantities ρ and μ.

We consider a system of initial perturbations that are kinematically similar. Then each individual perturbed state can be defined by assigning a length and time scales, selecting for this a certain char-

acteristic velocity u_0 and a certain characteristic quantity l_0 with the dimensions of length.

Therefore, the turbulent state of the fluid motion for a system of kinematically similar initial perturbations is defined by the following parameters:

$$\rho,\ \mu,\ l_0,\ u_0,\ t,\ x_1,\ x_2,\ x_3$$

where x_1, x_2, and x_3 are the coordinates of a point in space. (The reference frames are located similarly with respect to the distribution of the initial perturbations.)

The set of turbulent motions so obtained contains motions that are not dynamically similar. For two turbulent motions to be similar, it is necessary and sufficient that the Reynolds number should have the same value in both motions:

$$\frac{u_{01}l_{01}\rho_1}{\mu_1} = \frac{u_{02}l_{02}\rho_2}{\mu_2}$$

The instants of time and the coordinates of the points corresponding to similar states are determined from the relations

$$\frac{u_{01}t_1}{l_{01}} = \frac{u_{02}t_2}{l_{02}}, \qquad \frac{x_{i1}}{l_{01}} = \frac{x_{i2}}{l_{02}} \quad (i = 1, 2, 3)$$

The values of u and l are determined by functions such as

$$\left.\begin{aligned} \frac{u}{u_0} &= f\left(\frac{u_0 t}{l_0}, \frac{u_0 l_0 \rho}{\mu}, \frac{x_1}{l_0}, \frac{x_2}{l_0}, \frac{x_3}{l_0}\right) \\ \frac{l}{l_0} &= \varphi\left(\frac{u_0 t}{l_0}, \frac{u_0 l_0 \rho}{\mu}, \frac{x_1}{l_0}, \frac{x_2}{l_0}, \frac{x_3}{l_0}\right) \end{aligned}\right\} \tag{4.1}$$

In general, formulas of a similar kind will hold for any dimensionless mechanical quantity evaluated at one point of a fluid. These functions depend on dimensionless parameters defining the distributions of the initial perturbations as well as on the parameters mentioned above.

The quantities depending on the state of motion at two or more points must be considered when studying turbulent motion; these quantities can depend on the coordinates of several points. For example, the mean values of the products of the velocity components at m points $M_1\,(x_1', x_2', x_3')$, $M_2\,(x_1'', x_2'', x_3'')$, ..., $M_m\,(x_1^m, x_2^m, x_3^m)$

$$\tau_{k_1 k_2 \ldots k_n}(M_r, M_s, \ldots, M_d) = \overline{u_{k_1}(M_r)\, u_{k_2}(M_s) \ldots u_{k_n}(M_d)} \tag{4.2}$$
$$(k_1, k_2, \ldots, k_n = 1, 2, 3)$$

generate a tensor whose components depend on $3m$ coordinates x_i^k ($i = 1, 2, 3$; $k = 1, 2, \ldots, m$) for a system of similar initial perturbations. The subscripts $r, s, \ldots, d$ form a sequence of integers $1, 2, \ldots, m$ ($m \leqslant n$). In the general case, τ is essentially a function of all coordinates.

A turbulent flow is called homogeneous if all the mean quantities at each point are independent of the position of the point and if the mean values for the quantities depending on the positions of several points only depend on their relative location, i.e. only on the differences of the coordinates $x_i^p - x_i^q$. Functions (4.1) in a homogeneous turbulent velocity field are independent of x_1/l_0, x_2/l_0, and x_3/l_0. The averaged characteristics of the fluid motion near any two points are identical. Evidently, the initial perturbations in the case of a homogeneous turbulent flow must be uniformly distributed over the volume occupied by the fluid.

A homogeneous turbulent flow is called isotropic if the tensor relations formed from the velocity components, defined by (4.2), for arbitrary n and m, are independent of the orientation of the polyhedron $M_1M_2 \ldots M_m$ in space and are conserved under the transfer to the mirror images of this polyhedron in the coordinate planes. (If this condition is valid only for $2 \leqslant n < N$ and $2 \leqslant m < M$, where N and M are certain integers, then such a flow can be considered to be approximately isotropic.)

The components of the tensor relation depend on the positions of the coordinate axes relative to the polyhedron $M_1M_2 \ldots M_m$ and on the choice of the positive direction along the axes. The quantities $\tau_{k_1k_2 \ldots k_n}$, for an isotropic turbulent flow, have the same value in all coordinate systems oriented identically relative to various positions of the same polyhedron.

By definition, isotropic turbulent motion has the property of symmetry in the mean. It is assumed that all velocity directions are equally probable at each point of space during the sufficiently long time interval over which the average is taken; this is combined with the assumption that the fluid motion at each instant is continuous and has approximately the same velocity at neighbouring points.

Homogeneous isotropic turbulence can be considered the simplest kind of turbulent motion. A perturbed fluid is moved by inertia; internal viscous forces lead to the dissipation of kinetic energy so that the motion is characterized by damping and the decay of turbulent perturbations. The fundamental problem in the study of isotropic turbulence is the determination of the damping law. (At the present time, the existence and the possible different kinds of the isotropic turbulent motion of a viscous fluid still wait for a thorough theoretical analysis.)

Turbulent motions are characterized in the general case by the equalization (i.e. diffusion) of perturbations. Isotropic turbulent motion can be regarded in some cases as a sort of limiting turbulent motion, just as an unsteady flow can often be replaced approximately by a limiting steady flow.

Experiments show that the turbulent motion of air behind rapidly moving small-cell grids can be considered, at not too small and not

too large distances from them where the effect of flow boundaries is negligible, a practically homogeneous isotropic turbulent motion. The turbulent motion of air blown through a fixed grid also reduces to this case if a constant-velocity translation is added to the whole system. We encounter such a problem in investigating turbulence in wind-tunnels.

3. Symmetry of the Velocity Correlation Tensors. The isotropy assumption leads to a number of relations between the components of the tensor $\tau_{k_1k_2\ldots k_n}$. For example, if the components of the tensor $\tau_{k_1k_2\ldots k_n}$ are formed of the mean values of the velocity components of the common point $M_1 = M_2 = \ldots = M_m$, then, obviously, we shall have, for odd values of n,

$$\tau_{k_1k_2\ldots k_n} = 0$$

The only nonvanishing components are those in which each velocity component occurs to an even power, for even values of n. In particular, we have for $n = 1$:

$$\overline{u_1} = \overline{u_2} = \overline{u_3} = 0$$

for $n = 2$, we shall have

$$\overline{u_1u_2} = \overline{u_1u_3} = \overline{u_2u_3} = 0, \quad \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2} = \frac{1}{3}\,\overline{\mathfrak{b}^2} \tag{4.3}$$

and for $n = 3$, all the components are zero, etc.

Certain symmetry conditions hold if the points $M_1, M_2, \ldots, M_m$ are different, in particular, if we have just two points. In the latter case, the correlation tensor depends only on the distance r between the points in question. The components of $\tau_{k_1k_2\ldots k_n}$ depend on r and on the orientation of the coordinate axes relative to the segment M_1M_2.

It is easily verified that if the motion is isotropic, the following relations hold:

$$\overline{u_i(M_1)\,u_j(M_2)} = \overline{u_i(M_2)\,u_j(M_1)} \quad (i = 1, 2, 3; \quad j = 1, 2, 3) \tag{4.4}$$

Let us take the point M_1 at the origin and the point M_2 on the x_1-axis; then we have

$$\tau_{11} = b_d^d \neq 0, \quad \tau_{22} = \tau_{33} = b_n^n \neq 0, \quad \tau_{12} = \tau_{13} = \tau_{23} = 0 \tag{4.5}$$

The superscripts refer to the point M_2 and the subscripts to the point M_1. It is evident that the quantities b_d^d and b_n^n depend on x_1 and t and are even functions of x_1.

If the position of the point M_2 is arbitrary with respect to the coordinate axes, then all the τ_{ik} components can easily be expressed in terms of b_d^d and b_n^n. The formulas expressing τ_{ik} in terms of b_d^d and b_n^n are found from the formulas for the tensor components when the special coordinate system, in which the x_1-axis passes through

the point M_2, is transformed to an arbitrarily assigned coordinate system. These formulas are

$$\tau_{ik} = (b_d^d - b_n^n)\, l_{i1} l_{k1} + b_n^n \delta_{ik} \left(\delta_{ik} \begin{matrix} = 1, & i = k \\ = 0, & i \neq k \end{matrix}\right) \tag{4.6}$$

where l_{ik} are the direction cosines of the axes of the assigned coordinate system with respect to the specially chosen system.

We have for $r = 0$:

$$b_d^d = b_n^n = \frac{1}{3}\overline{\mathfrak{b}^2} = b \tag{4.7}$$

Now, let us consider the symmetry conditions satisfied by the third-order correlation tensor. The transformation formulas of the third-order tensor components for a transformation from one coordinate system to another are

$$\Pi'_{ijk} = \sum_{\alpha, \beta, \gamma} \Pi_{\alpha\beta\gamma} l_{\alpha i} l_{\beta j} l_{\gamma k} \tag{4.8}$$

where l_{sm} are the direction cosines of the new coordinate system. Let us take the transformation

$$x'_1 = -x_1, \quad x'_2 = x_2, \quad x'_3 = x_3$$

that reduces simply to a reversal in the direction of the x_1-axis. We have in this case

$$l_{11} = -1, \quad l_{22} = l_{33} = 1, \quad l_{sm} = 0 \quad (s \neq m)$$

consequently, if the subscript 1 occurs among the subscripts i, j, and k an odd number of times, it follows from (4.8) that

$$\Pi'_{ijk} = -\Pi_{ijk}$$

Let the point M_1 coincide with the origin and let the point M_2 lie on the x_1-axis. It then follows from the isotropy properties that the components of the third-order correlation tensor formed for the velocity components of the points M_1 and M_2 are independent of the directions of the x_2- and x_3-axes.

Hence it follows that the components containing the subscripts 2 or 3 an odd number of times are zero since they cannot change sign with a change in the direction of the x_2- or x_3-axis.

Hence, if the point M_2 lies on the x_1-axis, then the correlation tensor (formed from two velocity components at the point M_1 and one velocity component at the point M_2) has the following five non-vanishing components:

$$\tau_{111} = b_{dd}^d$$

$$\tau_{122} = \tau_{133} = b_{dn}^n$$

$$\tau_{221} = \tau_{331} = b_{nn}^d$$

The third-order correlation tensor components in any coordinate system can be expressed in terms of b^d_{dd}, b^n_{dn}, and b^d_{nn} by using (4.8). (The general symmetry theory readily yields the corresponding formulas for the components of the fourth- and higher-order tensors of moments in the case of isotropic turbulence as well as in some other symmetry situations (e.g. axial symmetry, etc.), see [15].) In the general case, the distance between the M_1 and M_2 points, which we shall subsequently denote by the letter r, must be taken instead of the variable x_1. As the distance between the points increases, their velocities become more and more independent statistically; consequently, the components of the velocity correlation tensors must approach zero as r tends to infinity.

The points M_1 and M_2 coincide at $r = 0$; in this case, we have

$$b^d_{dd} = b^n_{dn} = b^d_{nn} = 0$$

Interchanging the roles of the points M_1 and M_2 is equivalent to reversing the direction of the coordinate axes. Hence, it follows that

$$\tau_{ijk}(M_1,\ M_1,\ M_2) = -\tau_{ijk}(M_2,\ M_2,\ M_1) \tag{4.9}$$

These relations can be written in alternate form:

$$\left.\begin{aligned} b^d_{dd} &= -b^{dd}_d \\ b^n_{dn} &= -b^{dn}_n \\ b^d_{nn} &= -b^{nn}_d \end{aligned}\right\} \tag{4.10}$$

Therefore, the correlation moments b^d_{dd}, b^n_{dn}, and b^d_{nn} are odd functions of x_1 or r.

If the fluctuating velocity components are regular functions of the coordinates expanded in a Taylor series, then it is evident that the correlation moments can also be expanded in a Taylor series in r. The Taylor series for the second-order moments b^d_d and b^n_n will contain only the even powers of r, and the series for the third-order moments b^d_{dd}, b^n_{dn}, and b^d_{nn} will contain only the odd powers of r.

We shall show that the series for b^d_{dd} does not contain a linear term in r. Indeed, we have

$$b^d_{dd} = \overline{u_1^2(0,0,0)\,u_1(r,0,0)} = \overline{u_1^2\left(\frac{\partial u_1}{\partial r}\right)_{r=0}}\,r + \frac{1}{6}\,\overline{u_1^2\left(\frac{\partial^2 u_1}{\partial r^2}\right)_{r=0}}\,r^3 + \dots$$

The first term becomes zero since

$$\overline{u_1^2\left(\frac{\partial u_1}{\partial r}\right)_{r=0}} = \frac{1}{3}\left[\frac{\partial \overline{u_1^3}}{\partial r}\right]_{r=0}$$

and $\overline{u_1^3}\,(r) = 0$ because of the isotropy of the turbulent motion. Thus, the power series expansion of the moment b^d_{dd} must start with a term at least of order r^3.

4. Incompressibility Conditions and Dynamic Relations. Using the averaging operation, we can establish the relations between the independent components of the velocity correlation tensors from the incompressibility and the Navier-Stokes equations [16, 17]. The following relations are obtained:

$$b_n^n = b_d^d + \frac{r}{2}\frac{\partial b_d^d}{\partial r} \tag{4.11}$$

$$b_n^{nd} = -b_d^{nn} - \frac{r}{2}\frac{\partial b_d^{nn}}{\partial r} \tag{4.12}$$

$$b_d^{dd} = -2b_d^{nn} \tag{4.13}$$

The correlation moments b_d^d and b_n^n can be measured directly and independently in experiments. Using Simmons' experimental results, obtained in a wind-tunnel, Taylor [18] (1937) showed that the experiments are in a very good agreement with (4.11) (Fig. 21).

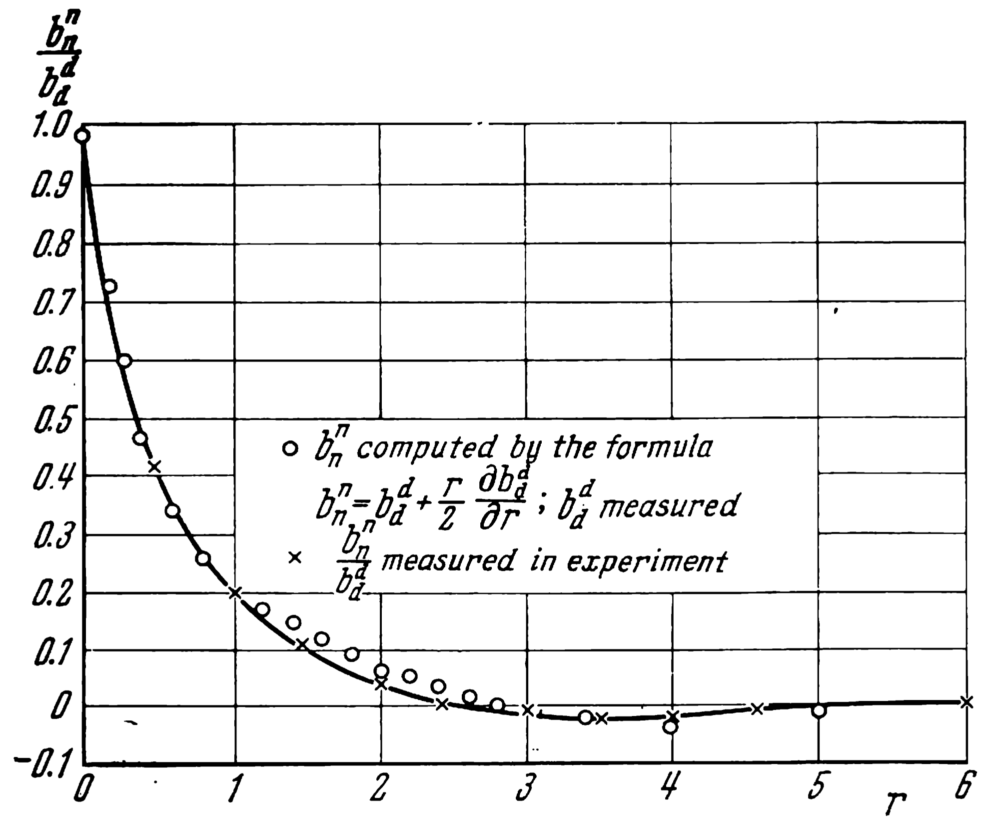

Fig. 21. Experimental data in good agreement with the theoretical formula relating b_d^d and b_n^n.

Relations (4.12) and (4.13) show that the power series in r for b_d^{nn} and b_n^{nd} must start with a term at least of order r^3 for small r since the series for b_d^{dd} has this property. Equations (4.11), (4.12), and (4.13) show that the determination of the velocity component correlation tensors of the second and the third order reduces to the determination of two functions $b_d^d\,(r,\ t)$ and $b_d^{nn}\,(r,\ t)$. Only one equation is obtained for these quantities from the Navier-Stokes equations,

namely,

$$\nu\left(\frac{\partial^2 b_d^d}{\partial r^2}+\frac{4}{r}\frac{\partial b_d^d}{\partial r}\right)-\frac{1}{2}\frac{\partial b_d^d}{\partial t}=\frac{\partial b_d^{nn}}{\partial r}+\frac{4}{r}b_d^{nn} \tag{4.14}$$

where $\nu=\mu/\rho$. This equation was obtained by Kármán and Howarth in another form for the functions $f=b_d^d/b$ and $h=b_d^{nn}/b^{3/2}$. Loitsyansky analysed (4.14) in the form given above [19].

Multiplying (4.14) by r^4, we obtain:

$$\frac{1}{2}\frac{\partial b_d^d r^4}{\partial t}=\frac{\partial}{\partial r}\left[\nu r^4\frac{\partial b_d^d}{\partial r}-r^4 b_d^{nn}\right] \tag{4.14'}$$

Assuming that for $r\to+\infty$ the order to which the functions $\partial b_d^d/\partial r$ and b_d^{nn} vanish is higher than $1/r^4$ and that the expression in the brackets becomes zero at $r=0$, we find

$$\frac{d}{dt}\int_0^\infty b_d^d r^4\,dr=0$$

or

$$\Lambda=\int_0^\infty b_d^d r^4\,dr=\text{const} \tag{4.15}$$

The existence of the invariant quantity Λ thus defined was discovered by Loitsyansky [21]. However, we may note that the properties of boundedness and invariance of Λ are established on the assumption about the order to which b_d^{nn} and the derivative $\partial b_d^d/\partial r$ vanish.

Equation (4.14) alone is inadequate to determine b_d^d and b_d^{nn}. The transition to the fourth- and higher-order correlation moments does not yield a closed system of equations either. The equation for the third-order moments contains the fourth-order moments; the next equations contain the fifth-order moments, etc. Besides these equations, we have also to assign initial conditions; consequently, additional hypotheses of a mechanical nature are required for a theoretical study of isotropic turbulence [20].

Similar conclusions were reached by Kármán and Howarth [17] in analysing the cases of small and large Reynolds numbers.

5. Concluding Stages in the Decay of Isotropic Turbulence. When the turbulent motion of a fluid is damped, the quantity $b=\overline{u_1^2}$ approaches zero as t tends to infinity.

The components of the third-order correlation moments are small in comparison with the components of the second-order correlation moments when the fluctuating velocities are very small. This affords a basis for neglecting the third-order moments in (4.14). Replacing

the right-hand side of (4.14) by zero, we obtain

$$\nu\left(\frac{\partial^2 b_d^d}{\partial r^2}+\frac{4}{r}\frac{\partial b_d^d}{\partial r}\right)-\frac{1}{2}\frac{\partial b_d^d}{\partial t}=0 \tag{4.16}$$

To find a particular solution of (4.16), the function $b_d^d(r, 0)$, which gives the initial distribution of the correlation moments of the longitudinal velocities of two points, must be known. We propose to obtain the solution of (4.16) for large values of the time t. The singularities of the function $b_d^d(r, 0)$ play a secondary role for large t; consequently, the initial conditions can be put into a simplified form when investigating the asymptotic behaviour of the function $b_d^d(r, t)$ as $t \to +\infty$.

Since the dimensions of b_d^d and r are different, the function $b_d^d(r, 0)$ involves dimensional constants as well as the distance r.

Now let us assume that the effect of the initial distribution of the correlation moments is represented in the asymptotic behaviour of b_d^d as $t \to +\infty$ by a single constant factor A with the dimensions $\mathrm{L}^p\mathrm{T}^q$. This constant represents the total properties of the initial distribution of the correlation moment b_d^d. In particular, the value of the constant A can be determined from the formula

$$A=\left[\int_0^\infty b_d^d(r, t)\frac{r^{p+2q+1}}{\nu^{2+q}}\Phi\left(\frac{r^2}{\nu t}\right)dr\right]_{t=0}$$

where $\Phi(r^2/(\nu t))$ is a certain function.

Let us put

$$b_d^d=A\nu^{1-p/2}\frac{f(r, t, \nu)}{t^{1+q+p/2}}$$

By hypothesis, the dimensionless quantity $f(r, t, \nu)$ depends only on the three dimensional parameters r, t, and ν; consequently, the function $f(r, t, \nu)$ depends only on the combination $\xi = r^2/(\nu t)$. We can therefore write

$$b_d^d=\frac{A\nu^{1-p/2}}{t^{1+q+p/2}}f\left(\frac{r^2}{\nu t}\right) \tag{4.17}$$

Later, we shall consider such motions for which $f(\infty) = 0$ and $f(0) = 1$; the first condition corresponds to the statistical independence of the velocities of two points as $r \to +\infty$ and the second condition is easily satisfied by a choice of the numerical value of the constant A.

We have from (4.17) for $r = 0$:

$$b=\frac{1}{3}\overline{\mathfrak{v}^2}=\frac{A\nu^{1-p/2}}{t^{1+q+p/2}} \tag{4.18}$$

Because of the damping of the turbulent motion, we must have $1 + q + p/2 > 0$; consequently, since $f(0) = 1$, the distribution of

the perturbations defined by (4.17) is characterized by irregularity at $t = 0$.

It is evident that the function

$$f\left(\frac{r^2}{\nu t}\right) = \frac{\overline{u_1(0, 0, 0, t)\, u_1(r, 0, 0, t)}}{\overline{u_1^2}}$$

is the correlation coefficient between the projections of velocities of two points onto the segment connecting these points. The correlation coefficient equals zero in the case under consideration at $t = 0$ if $r \neq 0$, and equals unity if $r = 0$.

Using (4.17), we obtain from (4.16)

$$f'' + \left(\frac{1}{8} + \frac{5}{2\xi}\right) f' + \frac{5\alpha}{4\xi} f = 0 \tag{4.19}$$

where

$$\alpha = \frac{1 + q + p/2}{10}$$

This equation has been obtained by Kármán and Howarth directly from the assumption that the correlation coefficient depends only on the combination $r^2/(\nu t)$. The number α is an arbitrary constant introduced by Kármán.

The solution of (4.19) depends only on the constant α.

For α = const, the dimensions of the constant A are fixed solely by the difference between the values of p and q. Formula (4.17) shows that the constant $A\nu^{1-p/2}$ must have the dimensions

$$L^2T^{q+p/2-1} = L^2T^{10\alpha-2}$$

The general solution of (4.19) is regular for all $\xi \neq 0, \infty$. The regular solution at $\xi = 0$ which satisfies the condition $f(0) = 1$ is

$$f(\xi) = M\left(10\alpha, \frac{5}{2}, -\frac{\xi}{8}\right)$$
$$= 1 - \alpha\xi + \frac{\alpha(10\alpha+1)}{4\cdot 7\cdot 2!}\xi^2 - \frac{\alpha(10\alpha+1)(10\alpha+2)}{4^2\cdot 7\cdot 9\cdot 3!}\xi^3 + \dots \tag{4.20}$$

where $M(\alpha, \gamma, x)$ is a confluent hypergeometric function [22].

The asymptotic expansion

$$f(\xi) = \frac{\Gamma\left(\frac{5}{2}\right)8^{10\alpha}}{\Gamma\left(\frac{5}{2} - 10\alpha\right)} \frac{1}{\xi^{10\alpha}} \left[1 + 8\frac{10\alpha\left(10\alpha - \frac{3}{2}\right)}{\xi}\right.$$
$$\left. + 8^2\frac{10\alpha(10\alpha+1)\left(10\alpha - \frac{3}{2}\right)\left(10\alpha - \frac{1}{2}\right)}{2!\xi^2} + \dots\right] \tag{4.21}$$

where Γ is the Euler gamma function, gives the solution for very large values of $\xi \to +\infty$.

The behaviour of the moment b_d^d as $r \to +\infty$ or as $t \to 0$ is easily explained by using (4.17) and (4.21). For all $\alpha > 0$, we have

$$\lim_{\frac{r^2}{\nu t} \to \infty} b_d^d r^{20\alpha} = A\nu^{2+q} \frac{\Gamma\left(\frac{5}{2}\right) 8^{10\alpha}}{\Gamma\left(\frac{5}{2} - 10\alpha\right)} \tag{4.22}$$

If

$$10\alpha - \frac{5}{2} = k$$

i.e. if

$$\alpha = \frac{1}{4} + 0.1k$$

where k is a positive integer, then the required solution of (4.19) finally takes the simple form

$$f = M\left(k + \frac{5}{2}, \frac{5}{2}, -\frac{\xi}{8}\right) = \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(k + \frac{5}{2}\right)} \left\{\frac{d^k}{d\lambda^k}\left[\lambda^{k+\frac{3}{2}} \exp\left(-\frac{\xi\lambda}{8}\right)\right]\right\}_{\lambda=1}$$

$$= \Gamma\left(\frac{5}{2}\right)\left[\frac{1}{\Gamma\left(\frac{5}{2}\right)} - \frac{k}{\Gamma\left(\frac{7}{2}\right)}\frac{\xi}{8} + \frac{k(k-1)}{\Gamma\left(\frac{9}{2}\right)2!}\frac{\xi^2}{8^2} - \dots\right.$$

$$\left.\dots + (-1)^k \frac{\xi^k}{\Gamma\left(k + \frac{5}{2}\right)8^k}\right] \exp\left(-\frac{\xi}{8}\right)$$

In this case, the distribution of the moment b_d^d for $t = 0$ is a source-like function. We have $b_d^d = 0$ for $r \neq 0$ and $t = 0$, and $b_d^d \to \infty$ for $r = 0$ as $t \to 0$.

Equation (4.16) can be interpreted as the heat conduction equation in a five-dimensional space with the symmetry relative to the origin. The solution corresponding to $\alpha = 1/4$ (when $k = 0$) can be considered the analogue of a heat source in a five-dimensional space [13, 19]. In this case, the solution is

$$b_d^d = A\nu^{1-p/2} \frac{\exp\left(-\frac{r^2}{8\nu t}\right)}{\sqrt{t^5}} \tag{4.23}$$

The constant $A\nu^{1-p/2}$ for $\alpha = 1/4$ has the dimensions $L^2T^{0.5}$.

It is easy to verify that the above solutions for

$$\alpha = \frac{1}{4} + 0.1k$$

where k is a positive integer, can be obtained from solution (4.23) or a simple source by differentiation with respect to time, namely

$$b_d^d = \frac{A\nu^{1-p/2}}{(-1)^k \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2} \cdots \left[\frac{5}{2} + (k-1)\right]} \frac{\partial^k}{\partial t^k}\left[\frac{\exp\left(-\frac{r^2}{8\nu t}\right)}{t^{5/2}}\right] \tag{4.24}$$

Clearly, this is the solution for an unsteady dipole of order k. The behaviour of the correlation coefficients $f(r/\sqrt{\nu t})$ in this case is shown in Fig. 22.

It can be shown that the parameter Λ is finite and nonzero at $\alpha = 1/4$.

From (4.17), we obtain

$$\Lambda = \int_0^\infty b_d^d r^4\, dr = \frac{A\nu^{1-p/2}}{2} \frac{(\nu t)^{5/2}}{t^{10\alpha}} \int_0^\infty f(\xi)\, \xi^{3/2}\, d\xi \tag{4.25}$$

This equation shows that constant values of Λ other than zero or ∞ are incompatible with the inequality $\alpha \neq 1/4$. Expansion (4.21) shows that $\Lambda = \infty$ for $0 < \alpha < 1/4$. We have $\Lambda = 0$ for $\alpha > 1/4$; in this case, $f(\xi)$ changes sign as ξ changes from zero to infinity.

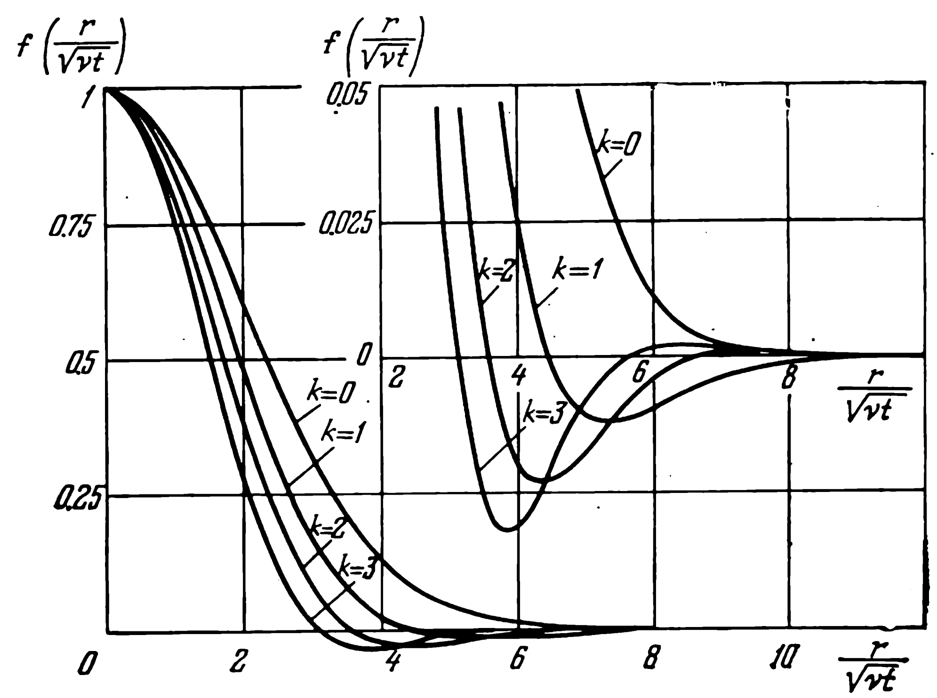

Fig. 22. Correlation coefficient for source-type motion with different values of k.

The solutions for $b_d^d(r, t_0)$ determined by (4.17) and (4.20) are continuous functions for all α when $t_0 > 0$. Clearly, in these particular cases and, therefore, in the general case the decay law is strongly influenced by the properties of the initial perturbations. *Consequently, in order to obtain asymptotic damping laws from the solutions considered we require either additional hypotheses of a mechanical character or relevant experimental data.*

6. Problem of Turbulent Motion in a Wind-Tunnel. We have already pointed out that the investigation of isotropic turbulence is related to the study of turbulence produced by the guide vanes of wind-tunnels.

We shall now discuss the development of turbulent motion of an incompressible fluid behind a grating which moves forward at a

constant velocity u along the x-axis. For simplicity, we assume that the fluid is infinite in extent and that the grating is formed by a doubly periodic system of congruent cells arranged forward in a plane perpendicular to the x-axis. We consider the family of motions behind gratings of geometrically fixed shape.

The fluid motion in a plane perpendicular to the x-axis is determined by the system of parameters

$$\rho,\ \mu,\ u,\ M,\ x = u\,(t - t_0)$$

where M is the characteristic grating dimension, and x is the coordinate of the plane in question; the constant t_0 is determined by the origin for x.

The dimensionless quantities of the motion depend on the two parameters

$$\frac{x}{M} \text{ and } \frac{\rho u M}{\mu}$$

(The flow in experiments with turbulent fluid motion behind gratings is not infinite, so that the channel or jet shape may affect some characteristics of fluid motion in a different manner and, in particular, via cross-sectional dimension L or, correspondingly, the number $\rho uL/\mu$.) We assume that the turbulent motion is isotropic for large enough x/M and that the development of isotropic turbulence in different planes perpendicular to the x-axis only differs in phase. In this case, the characteristics of the turbulent fluid motion are determined by the parameters ρ, μ, u, M, and t.

The correlation coefficients $f = b_d^d/b$ and $h = b_d^{nn}/b^{3/2}$ depend on the dimensionless parameters

$$\frac{\rho u M}{\mu},\quad \frac{ut}{M},\quad \frac{r^2}{\nu t}$$

Formula (4.17) is obtained on the assumption that the effect of the parameter ut/M becomes negligible for its sufficiently large values.

It follows from (4.18) that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{b_0}}\left(\frac{t}{t_0}\right)^{5\alpha}$$

Putting $t = t_0 + x/u$, we obtain

$$\frac{u}{\sqrt{b}} = \frac{u}{\sqrt{b_0}}\left(1 + \frac{x}{ut_0}\right)^{5\alpha} \tag{4.26}$$

Formula (4.26) gives the decay law of turbulent fluctuations along the axis of a tube.

Taylor [23] proposed the empirical formula

$$\frac{u}{\sqrt{b}} = A + B\,\frac{x}{M} \tag{4.27}$$

where A and B are constants, based on a number of experimental data obtained in wind-tunnels.

To ensure agreement between (4.26) and (4.27) we must put $\alpha = 1/5$. We have $[A\nu^{1-p/2}] = L^2$ for $\alpha = 1/5$.

7. Turbulent Motions with Large Fluctuations. If the parameter ut/M can be neglected, then the correlation coefficients remain constant for

$$\chi = \frac{r}{\sqrt{\nu t}} = \frac{r}{l} = \text{const}$$

The effect of time reduces to rescaling for r.

The rescaling in the solutions considered above is determined by the relation

$$l = \sqrt{\nu t} = \sqrt{\frac{\nu (x + x_0)}{u}} \tag{4.28}$$

It follows from formula (4.28) that the scale l is constant in time at $x = \text{const}$, i.e. at the fixed point relative to the grating. At the same time (4.28) shows that this scale is dependent on the velocity u.

Taylor [23] gives certain experimental data that do not support the last conclusion; because of this it was necessary to modify and improve the theory dealing with large fluctuations. The exchange of momentum between masses of displaced fluid is very important for large fluctuations. The fluid inertia property plays a leading part in these processes.

Viscosity plays a large part in the development of motions with very small fluctuations when the basic process is one of the kinetic energy dissipation. To study large fluctuations, we assume, following Kármán and Howarth [17], that for sufficiently large values of r the relations

$$\frac{b_d^d}{b} = f\left(\frac{r}{l}\right) \tag{4.29}$$

$$\frac{b_d^{nn}}{b^{3/2}} = h\left(\frac{r}{l}\right) \tag{4.30}$$

where l is a linear function of time and of constant parameters defining isotropic turbulent motion. Formulas (4.29) and (4.30) show that the effect of time on the correlation coefficients reduces to rescaling for the distance l. Viscosity affects the energy dissipation implicitly via the quantities b and l.

(The conclusions that follow, presented in the first, 1944, edition of this monograph (see also [21]), are valid as long as the above-formulated assumptions hold (only for two-point moments of the second and third orders see p. 123): the homogeneity and isotropy of the turbulent motion of an incompressible fluid, with ensuing equation (4.14), and assumptions included in formulas (4.29) and (4.30).

The question whether hypotheses (4.29) and (4.30) agree with experiments remained unanswered for thirty years. Doubts were frequently expressed about the adequacy of these hypotheses in describing the experimental results. Recently Korneev [24] demonstrated, after a painstaking processing of published experimental data, that, with a suitable choice of constants appearing in the mathematical solution, assumptions (4.29) and (4.30) provide both a qualitative and a quantitative fits to the experiments.

Korneev also studied [24] analogous asymptotic relations describing the decay of homogeneous isotropic turbulence under more general assumptions, namely, hypotheses of the kind

$$b_d^d = b_1(t) - b_2(t)\,\beta_2\left(\frac{r}{l}\right),\ b_d^{nn} = b_3^{3/2}\beta_3\left(\frac{r}{l}\right)$$

satisfied only in the range $0 \leqslant \chi_1 \leqslant r/l \leqslant \chi_2 \leqslant \infty$, where χ_1 and χ_2 are limiting bounds. In this interval these more general assumptions are identical to (4.29) and (4.30) if b_2/b_1 = const and b_3/b_1 = = const.

Korneev emphasized that the theory remains mostly valid even when hypotheses (4.29) and (4.30) are made less strong by assuming that they hold only in the range $0 \leqslant \chi_1 \leqslant \chi \leqslant \chi_2 \leqslant \infty$.)

We shall not attribute any specific geometric or physical meaning to l. The definition of l can be considered an additional hypothesis. In particular, *if it is assumed that the invariant* $\Lambda \neq 0,\ \infty$, that is, that an appropriate integral exists and the conditions formulated on p. 128 are satisfied, *then we obtain a relation between l and b.* Indeed,

$$\Lambda = b\int_0^\infty f\left(\frac{r}{l}\right) r^4\,dr = bl^5\int_0^\infty f(\chi)\,\chi^5\,d\chi$$

Hence, we find

$$l = a\left(\frac{\Lambda}{b}\right)^{1/5} \tag{4.31}$$

here a is a constant. If we still assume that $l = \sqrt{\nu t}$, as in the solution for small fluctuations analysed above, then we again obtain the decay law corresponding to $\alpha = 1/4$,

$$b = \frac{a^5\Lambda}{(\nu t)^{5/2}}$$

Using (4.29) and (4.30), equation (4.14) yields

$$h'(\chi) + \frac{4h(\chi)}{\chi} - \frac{1}{2}\chi f'(\chi)\frac{1}{b^{1/2}}\frac{dl}{dt} + \frac{1}{2}f(\chi)\frac{l}{b^{3/2}}\frac{db}{dt} = \frac{\nu}{b^{1/2}l}\left(f''(\chi) + \frac{4f'(\chi)}{\chi}\right) \tag{4.32}$$

This relation can be satisfied by using various assumptions. Following Kármán and Howarth [17] we consider the motion corresponding to high values of the Reynolds number $\sqrt{b}\,l/\nu$ and, consequently, neglect the right-hand side of (4.32). Then we obtain

$$h' + \frac{4h'}{\chi} = \frac{1}{2}\chi f' kc + \frac{1}{2} fc \tag{4.33}$$

where

$$\frac{1}{\sqrt{b}}\frac{dl}{dt} = kc \tag{4.34}$$

$$\frac{1}{\sqrt{b^3}}\frac{dl}{dt} = -c \tag{4.35}$$

The dimensionless quantities kc and c are independent of χ. It follows from (4.33) and from the general properties of $f(\chi)$ that the quantities kc and c are constants.

Integrating (4.34) and (4.35), we obtain

$$\frac{l}{l_0} = \left(\frac{b_0}{b}\right)^k \tag{4.36}$$

We have for $k \geqslant -1/2$

$$\frac{b}{b_0} = \left(\frac{t_0+t}{2t_0}\right)^{-2/(2k+1)} \quad \text{and} \quad \frac{l}{l_0} = \left(\frac{t_0+t}{2t_0}\right)^{2k/(2k+1)} \tag{4.37}$$

where the time corresponding to the values of l_0 and b_0 is denoted by t_0. (The condition $k \geqslant -1/2$ must be satisfied to ensure that the mean velocity of the turbulent motion does not vanish at any finite instant of time.)

If we combine the assumption that an invariant $\Lambda \neq 0$ exists with formula (4.37), then it follows from (4.31) that $k = 1/5$. Substituting $k = 1/5$ into (4.37), we obtain Kolmogorov's result [25]

$$\frac{b}{b_0} = \left(\frac{t_0+t}{2t_0}\right)^{-10/7} \quad \text{and} \quad \frac{l}{l_0} = \left(\frac{t_0+t}{2t_0}\right)^{2/7} \tag{4.37'}$$

which he found by using a number of assumptions including (4.29) and (4.31). (It will be shown in item 8 that assumptions (4.29), (4.30), and (4.31) are mutually exclusive if $h \neq 0$, and if $h = 0$, laws are derived which differ from (4.37').)

The exponent k above is arbitrary. If we assume that for a wind-tunnel the scale l is constant and proportional to the dimensions of the grating cells M, that is, $l = \text{const}\cdot M$, then we find $k = 0$ and

$$\frac{u}{\sqrt{b}} = \frac{u}{2\sqrt{b_0}}\left(1+\frac{t}{t_0}\right) = A + B\frac{x}{M}$$

This formula agrees with the Taylor formula (4.27).

8. Laws of Decay of Turbulence with Account Taken of the Third-Order Moments. The laws of turbulent growth, represented by (4.37),

are based on the Kármán and Howarth additional assumption [17] that the right-hand side of (4.32) is zero. If the right-hand side of (4.32) is put equal to zero, we obtain for $f(\chi)$

$$f'' + \frac{4f}{\chi} = 0$$

which has no solution satisfying the condition $f(0) = 1$ except the solution $f(\chi) = 1$, and does not satisfy the physical condition $f(\infty) = 0$.

Therefore, the Kármán-Howarth solution is approximate; moreover, it does not enable us to determine the functions $f(\chi)$ and $h(\chi)$. Consequently, this solution is unsatisfactory.

Furthermore, we can find all physically admissible exact solutions of (4.14) *simply by using the two basic Kármán-Howarth assumptions included in the formulas*

$$\frac{b_d^d}{b} = f\left(\frac{r}{l}\right) \tag{4.29}$$

and

$$\frac{b_d^{nn}}{b^{3/2}} = h\left(\frac{r}{l}\right) \tag{4.30}$$

where l and b are certain functions of time.

Because of these assumptions, (4.14) reduces to (4.32). Now let us consider the exact solutions of this equation in more detail. (Several unknown functions enter into equation (4.14) and, correspondingly, equation (4.32) following from (4.14). At the first glance, not more than one function can be found from a single equation. Nevertheless, a careful consideration of the mathematical structure of this equation makes it possible to carry out an analysis of all possible cases and to find, to the accuracy of one basic constant α (see below), all admissible solutions of the problem in question. This aspect and the appropriate mathematical analysis of the problem escaped the attention of a number of scientists who developed the theory of turbulent motions in fluids and processed the experimental data.)

Differentiating (4.32) with respect to time for constant χ, we obtain

$$\frac{1}{2}\chi f'(\chi)\frac{d}{dt}\left(\frac{1}{b^{1/2}}\frac{dl}{dt}\right) - \frac{1}{2}f(\chi)\frac{d}{dt}\left(\frac{l}{b^{3/2}}\frac{db}{dt}\right) + \left(f''(\chi) + \frac{4f'(\chi)}{\chi}\right)\frac{d}{dt}\left(\frac{\nu}{\sqrt{b}\,l}\right) = 0 \tag{4.38}$$

We shall analyse the following possible cases associated with (4.38):

(1) The functions $\chi f'(\chi)$, $f(\chi)$, and $f''(\chi) + 4f'(\chi)/\chi$ are linearly independent.

(2) There exists only one independent linear relation with constant coefficients of the type

$$c_1\chi f' + c_2 f + c_3\left(f'' + \frac{4f'}{\chi}\right) = 0 \tag{4.39}$$

in which not all c_1, c_2, and c_3 equal zero.

(3) Two linearly independent relations with constant coefficients exist between the functions $\chi f'$, f, $f'' + (4f'/\chi)$. Since $f \not\equiv 0$, these relations can always be written in the form

$$f'\chi = c_1' f \text{ and } f'' + \frac{4f'}{\chi} = c_2' f$$

It is not difficult to see that in the last case

$$c_1' = c_2' = 0 \text{ and } f = \text{const}$$

This solution is of no physical interest, so the third case can be ignored.

In the first case, all the coefficients in (4.38) equal zero, consequently:

$$\frac{1}{\sqrt{b}}\frac{dl}{dt} = kc, \quad \frac{1}{\sqrt{b^3}}\frac{db}{dt} = -c, \quad \frac{\nu}{\sqrt{b}\, l} = \frac{1}{mc} \tag{4.40}$$

where k, c, and m are certain constants. System (4.40) gives

$$l = c\sqrt{m\nu(t+t_0)}, \quad b = \frac{m\nu}{t+t_0}, \quad k = \frac{1}{2} \tag{4.41}$$

Thus, we have obtained l and b as specific functions of time. These agree with (4.37) for $k = 1/2$. In this case, only one equation is obtained for the two functions $f(\chi)$ and $h(\chi)$. (It is evident that in this case the quantity Λ cannot be finite and cannot be identically zero, otherwise (4.31) would contradict (4.41).)

Now, let us investigate the second case. The coefficient c_3 in (4.39) is clearly not zero since, otherwise, the function $f(\chi)$ would have to satisfy the equation

$$c_1 f + c_2\chi f' = 0$$

This has no solution satisfying the condition $f(0) = 1$ for $c_1 \neq 0$; for $c_1 = 0$, we obtain $f' = 0$ or $f = \text{const} \neq 0$, in conflict with the condition $f(\infty) = 0$.

Since $c_3 \neq 0$, (4.39) can be written in the form

$$f'' + \frac{4}{\chi}f' + \frac{a_1}{2}\chi f' + \frac{a_2}{2}f = 0 \tag{4.42}$$

where a_1 and a_2 are constant coefficients.

We find from (4.38) and (4.42) that

$$\frac{1}{2}\chi f'\left[\frac{d}{dt}\frac{1}{\sqrt{b}}\frac{dl}{dt}-a_1\frac{d}{dt}\frac{\nu}{\sqrt{b}\,l}\right] -\frac{1}{2}f\left[\frac{d}{dt}\frac{l}{\sqrt{b^3}}\frac{db}{dt}+a_2\frac{d}{dt}\frac{\nu}{\sqrt{b}\,l}\right]=0$$

Since the functions $\chi f'$ and f are linearly independent, the expressions in the brackets must vanish so that

$$\frac{1}{\sqrt{b}}\frac{dl}{dt}=a_1\frac{\nu}{\sqrt{b}\,l}+p, \qquad \frac{l}{\sqrt{b^3}}\frac{db}{dt}=-a_2\frac{\nu}{\sqrt{b}\,l}+q \tag{4.43}$$

where p and q are constants of integration.

Substituting (4.43) into (4.32) and using (4.12), we find the equation determining $h(\chi)$

$$h'+\frac{4h}{\chi}-\frac{1}{2}\chi f'p+\frac{1}{2}fq=0 \tag{4.44}$$

We have already shown that the expansion of the function $b_d^{nn}/b^{3/2}=h$ in powers of $r/l=\chi$ must start with terms of order χ^3; consequently, it follows from (4.44) that $q=0$.

Hence, in the second case we have a complete system of equations (4.42), (4.43), and (4.44) to determine h and f as functions of the variable χ and the quantities l and b as functions of the time t.

These equations contain three arbitrary constants a_1, a_2, and p. One of these constants is clearly negligible. Indeed, a transformation $\chi=\lambda\chi'$, where λ is a constant, reduces to multiplying the still undefined scale factor l by a constant $1/\lambda$ ($l=l'/\lambda$). Carrying out this transformation in (4.42), (4.43), and (4.44), we obtain the same equations as before with transformed values a_1', a_2', and p' defined by

$$a_1'=a_1\lambda^2, \qquad a_2'=a_2\lambda^2, \qquad p'=p\lambda$$

Fixing a particular nonzero value for a_1 or for a_2 is equivalent to fixing the scale for l.

From the second equation in (4.43), it follows from the condition of damping of turbulent fluctuations ($b\to 0$ as $t\to\infty$) that $a_2>0$.

Furthermore, we also assume as a physical condition that $l\to+\infty$ as $t\to\infty$ or as $b\to 0$.

Equations (4.43) can be integrated to give the relations

$$\left.\begin{aligned}\frac{1}{l}&=\frac{2p}{\nu(a_2-2a_1)}\sqrt{b}+cb^{a_1/a_2}\\ \text{for}\quad a_2&-2a_1\neq 0\end{aligned}\right\} \tag{4.45}$$

and

$$\left.\begin{aligned}\frac{1}{l}&=\frac{p}{\nu}\sqrt{b}\,(\ln b-c_1)\\ a_2-2a_1&=0\end{aligned}\right\}\tag{4.46}$$

for

where c and c_1 are constants of integration.

It follows from (4.45), from the inequality $a_2 > 0$, and from the condition $l \to \infty$ as $b \to 0$ that $a_1 > 0$ if $c \neq 0$.

Let us consider the case when $c = 0$ and $a_1 = 0$. Equation (4.42) has a unique solution satisfying the condition $f(0) = 1$ for $a_1 = 0$. This solution is easily determined:

$$f(\chi)=3\left[\frac{\sin\sqrt{\frac{a_2}{2}}\chi}{\left(\sqrt{\frac{a_2}{2}}\chi\right)^3}-\frac{\cos\sqrt{\frac{a_2}{2}}\chi}{\frac{a_2}{2}\chi^2}\right]$$

This function oscillates as $\chi = r/l$ increases, has a finite number of zeros, and decreases slowly as $1/\chi^2$. On this basis, we shall not consider the case $a_1 = 0$ any further.

We are free to choose the parameter λ, so we put $a_1 = 1/2$; then (4.42) takes the form

$$f''+\left(\frac{4}{\chi}+\frac{\chi}{4}\right)f'+\frac{a_2}{2}f=0\tag{4.47}$$

Putting $a_2 = 10\alpha > 0$ and

$$\frac{r^2}{l^2}=\chi^2=\xi$$

transforms (4.47) to (4.19) investigated earlier in studying the damping of very small fluctuations. Therefore, we obtain the same solution for $f(\xi)$ as in the small fluctuations case, namely,

$$f(\xi)=M\left(10\alpha,\ \frac{5}{2},\ -\frac{\xi}{8}\right)\tag{4.48}$$

but now $l \neq \sqrt{\nu t}$ and $h \neq 0$.

Therefore, allowing for the third-order moments influences the correlation coefficient $f = b_d^d/b$ only by changing the dependence of the linear scale l on time.

rUsing (4.44) ($q = 0$), the function $h(\chi)$ is easily expressed in tems of the function $f(\chi)$. Integrating (4.44) and using the condition $h = 0$ when $\chi = 0$, we obtain

$$h=\frac{p}{2}\frac{1}{\chi^4}\int_0^{\chi}\chi^5 f'(\chi)\,d\chi\tag{4.49}$$

If $\alpha = 1/4$ and $h \neq 0$, then it is evident that the function $h(\chi)$ vanishes as $1/\chi^4$ when $\chi \to \infty$; consequently, the third-order moment b_d^{nn} is of order $1/r^4$ as $r \to +\infty$ and, therefore, the conditions necessary for the invariance of Λ (see p. 128) are not satisfied.

Hence, we have $\Lambda = \infty$ for $\alpha < 1/4$, $\Lambda = 0$ for $\alpha > 1/4$, and Λ is finite and nonzero but not constant for $\alpha = 1/4$ and $p \neq 0$ (the third-order moments different from zero). Indeed, we find from (4.14′) and (4.49), for $\alpha = 1/4$, that

$$\frac{d\Lambda}{dt} = -p \int_0^\infty \chi^5 f'(\chi)\, d\chi\, l^4 b^{3/2} \neq 0$$

If $p = 0$, then the integral of Λ is finite and invariant but the third-order moments are zero in this case.

Hence, if the third-order moments are nonzero and the hypotheses expressed by (4.29) *and* (4.30) *are fulfilled, then the integral of* Λ *is either zero or infinity and is therefore unsuitable to determine scales according to* (4.31), *or the quantity* Λ *varies with time for* $\alpha = 1/4$ *and, consequently, cannot be considered a characteristic constant.*

Multiplying (4.47) by $\chi^4\, d\chi$ and integrating, we find

$$\int_0^\chi \chi^5 f'\, d\chi = -4\chi^4 f' - 20\alpha \int_0^\chi \chi^4 f\, d\chi$$

$$= -4\chi^4 f' - 4\alpha\chi^5 f + 4\alpha \int_0^\chi \chi^5 f'\, d\chi$$

Using this relation for $\alpha \neq 1/4$, we obtain

$$h = \frac{2p}{4\alpha - 1}\,[f'(\chi) + \alpha\chi f(\chi)] \tag{4.50}$$

Equations (4.43) determine l and b as functions of time. If we put $a_1 = 1/2$, $a_2 = 10\alpha$, and $q = 0$, they become

$$\frac{1}{\sqrt{b}}\frac{dl}{dt} = \frac{1}{2}\frac{\nu}{\sqrt{b}\, l} + p, \qquad \frac{l}{\sqrt{b^3}}\frac{db}{dt} = -10\alpha\frac{\nu}{\sqrt{b}\, l} \tag{4.51}$$

In the particular case of $p = 0$, system of equations (4.51) and (4.47) evidently has the solution

$$l = \sqrt{\nu t}, \qquad b = \frac{A'}{(\nu t)^{10\alpha}}, \qquad f = M\left(10\alpha, \frac{5}{2}, -\frac{\xi}{8}\right), \quad \text{and} \quad h \equiv 0$$

which corresponds to the Kármán-Howarth small-fluctuations solution.

Let us introduce now Taylor's linear microscale $\lambda(t)$, frequently used in a number of papers devoted to isotropic turbulence, by the

formula

$$\frac{1}{\lambda^2} = -\frac{\partial^2 f(r, t)}{\partial r^2}\Big|_{r=0}$$

By using formula (4.48) for $f(\xi)$ ($\xi = r^2/l^2$) containing a known confluent function, we readily derive the following relation between l and λ:

$$l = \lambda\sqrt{\alpha} \tag{4.52}$$

Now consider the case of $p \neq 0$, rewriting formulas (4.45) and (4.46) by using λ for the scale instead of l and introducing the following notation:

$$\left.\begin{aligned} \lambda^* &= \frac{\nu}{|p|\sqrt{b^*}} > 0 \\ w &= \frac{b}{b^*}, \quad \mu = \frac{1-10\alpha}{20\alpha} \end{aligned}\right\} \tag{4.53}$$

Here the quantity $b^* > 0$ is defined for $\alpha \neq 0.1$ by the following formula derived from (4.45):

$$\left|\frac{c(a_2-2a_1)}{2p}\right| = \left|\frac{c(10\alpha-1)}{2p}\right| = b^{*^{-\mu}}$$

For $\alpha = 0.1$, the constant b^* is related to the constant c_1 in formula (4.46) by

$$c_1 = \ln b^*$$

It is suggested that we should use scale constants λ^* with length dimensions and b^* with velocity dimensions instead of the constants p and c when $\alpha \neq 0.1$ or p and c_1 when $\alpha = 0.1$.

The following relations can be obtained from (4.45):

when $\alpha \neq 0.1$, $p(10\alpha - 1) > 0$, $c > 0$, and $0 < w < \infty$ (this family of solutions will be denoted by α_+)

$$\frac{\lambda^*}{\lambda} = \frac{2\sqrt{\alpha_+ w}}{|10\alpha_+ - 1|}(1 + w^\mu) \tag{4.54}$$

when $\alpha \neq 0.1$, $c(10\alpha - 1) > 0$, $p < 0$, and $0 < w < 1$ (this family of solutions will be denoted by α_-)

$$\frac{\lambda^*}{\lambda} = \frac{2\sqrt{\alpha_- w}}{(1 - 10\alpha_-)}(1 - w^\mu) \tag{4.55}$$

when $\alpha = 0.1$, $p < 0$, and $0 < w < 1$, we find from (4.46) that

$$\frac{\lambda^*}{\lambda} = -\sqrt{\frac{w}{10}}\ln w \tag{4.56}$$

These are the relations between λ and w under conditions $\lambda^* > 0$, $\lambda > 0$, and $\lambda \to \infty$ as $w \to 0$, establishing inequalities $p(10\alpha - 1) >$

> 0 in (4.54) and $p < 0$ in (4.56) and (4.55), and fixing the above-indicated ranges of possible variation for $w = b/b^*$.

The second of equations (4.51) yields the following forms of the relation between t and w:

in case (4.54)

$$\tau = \frac{\nu (t + t^*)}{\lambda^{*2}} = \frac{(10\alpha_+ - 1)^2}{40\alpha_+} \int_w^\infty w^{-2} (1 + w^\mu)^{-2} \, dw \tag{4.57}$$

in case (4.55)

$$\tau = \frac{\nu (t + t^*)}{\lambda^{*2}} = \frac{(10\alpha_- - 1)^2}{40\alpha_-} \int_w^{w_0} w^{-2} (1 - w^\mu)^{-2} \, dw \tag{4.58}$$

and for $\alpha = 0.1$

$$\tau = \frac{\nu (t + t^*)}{\lambda^{*2}} = \int_w^{w_0} w^{-2} \ln^{-2} w \, dw \tag{4.59}$$

Here τ is the dimensionless time, and t^* is a constant of integration with time dimensions. The value of t^* in formulas (4.58) and (4.59) depends on the choice of w_0. To specify the problem, we assume that w_0 satisfies the condition

$$\left.\frac{d\lambda}{dw}\right|_{\tau=1} = \left.\frac{d\lambda}{d\tau}\right|_{\tau=1} = 0$$

that is, w_0 and $\tilde{w}$ are given by the equations

$$\int_{\tilde{w}}^{w_0} w^{-2} (1 \pm w^\mu)^{-2} \, dw = \frac{40\alpha}{(10\alpha - 1)^2}$$

and

$$\left|\frac{d\lambda}{dw}\right|_{\tilde{w}} = \left|\frac{d\lambda}{dt}\right|_{\tilde{w}} = 0$$

or

$$\int_{\tilde{w}}^{w_0} w^{-2} \ln^{-2} w \, dw = 1 \quad \text{and} \quad \left|\frac{d\lambda}{dw}\right|_{\tilde{w}} = \left|\frac{d\lambda}{dt}\right|_{\tilde{w}} = 0$$

The above-written systems of exact solutions thus depend on the four constant parameters α (α_+, or α_-, or $\alpha = 0.1$), b^*, λ^*, and t^*, with the last three acting as scale factors.

The family of dimensionless functions

$$f\left(\frac{r}{\lambda}\right), \quad \frac{1}{p} h\left(\frac{r}{\lambda}\right), \quad \frac{b}{b^*}(\tau), \quad \text{and} \quad \frac{\lambda}{\lambda^*}(\tau)$$

depends on a single dimensionless parameter α (α_+, or α_-, or $\alpha = 0.1$); according to (4.53), $p = \pm \nu / \lambda^* \sqrt{b^*}$. Consequently, the param-

eter α is indeed a single basic parameter for the solutions found. For both the families, the functions $f(r/\lambda)$ and $p^{-1}h(r/\lambda)$ are identical for $\alpha_+ = \alpha_-$, and the difference appears only for the functions

$$\frac{b}{b^*}(\tau, \alpha) \quad \text{and} \quad \frac{\lambda}{\lambda^*}(\tau, \alpha)$$

Let us consider also asymptotic functions for b/b^* and λ/λ^* as $\tau = \nu(t + t^*)/\lambda^2 \to \infty$. These formulas have identical form for the two families:

for $\alpha > 0.1$

$$\left.\begin{aligned} \frac{\lambda}{\lambda^*} &\simeq \frac{1}{\lambda^*}\sqrt{\frac{\nu}{\alpha}(t+t^*)} \\ \frac{b}{b^*} &\simeq \left[\frac{(10\alpha-1)^2\lambda^{*2}}{4\nu(t+t^*)}\right]^{10\alpha} \end{aligned}\right\} \tag{4.60}$$

for $\alpha < 0.1$

$$\left.\begin{aligned} \frac{\lambda}{\lambda^*} &\simeq \frac{1}{\lambda^*}\sqrt{10(t+t^*)\nu} \\ \frac{b}{b^*} &\simeq \frac{(10\alpha-1)^2\lambda^{*2}}{40\alpha\nu(t+t^*)} \end{aligned}\right\} \tag{4.61}$$

Figures 23-26 plot the results of calculations by the derived exact and asymptotic formulas for $\alpha = 0.05$, 0.08, 0.1, 0.15, and 0.2.

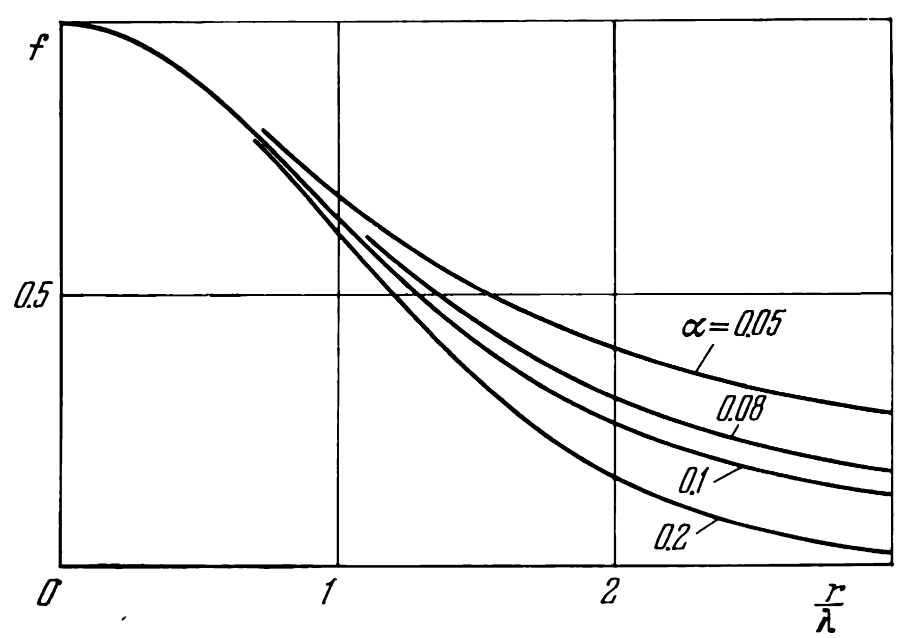

Fig. 23. The function $f(r/\lambda)$ plotted by using (4.20) for different values of α.

For rational values of α, the integrals in formulas (4.55) and (4.54) are found in elementary functions. The straight lines corresponding to asymptotic laws, transformable to power functions according to (4.57) and (4.58), are plotted by dashed lines in logarithmic scale in Figs. 25 and 26. These graphs show that the exact laws of decay of isotropic turbulence (derived with the weakest assumptions based on equalities (4.29) and (4.30)) tend to power functions as $t \to \infty$.

In the presented theory this is a corollary, and not an initial assumption.

As we find from formulas (4.17) and (4.18) for $\alpha > 0.1$, asymptotic laws (4.60), found with the third-order moments taken into account, are the same as when the third-order moments are neglected. If, however, $\alpha < 0.1$ (this is demonstrated below when α is found

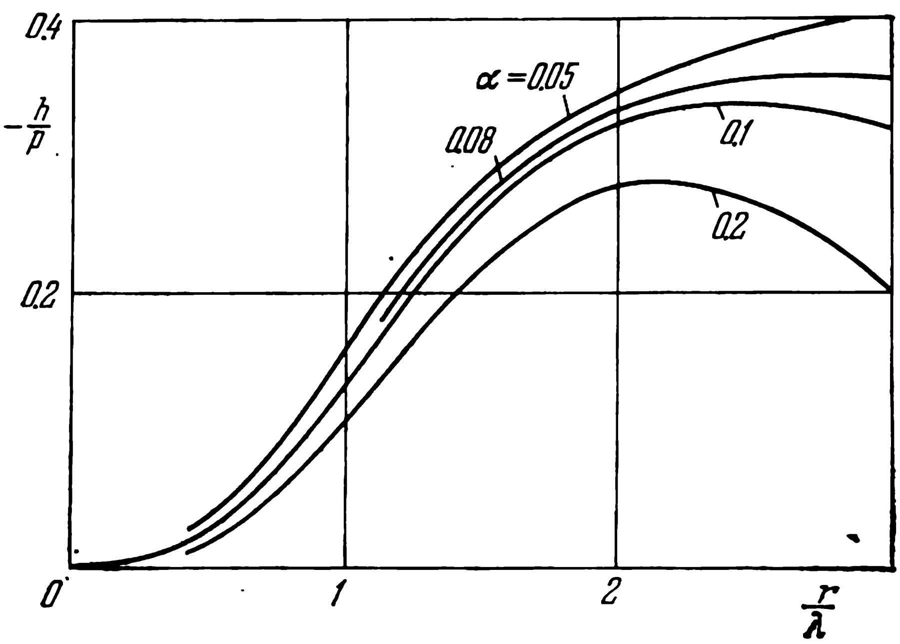

Fig. 24. The function $-\frac{1}{p}h\,(r/\lambda)$ plotted by using (4.50) for different values of α.

from experimental data), asymptotic laws (4.61) for b/b^* differ from relation (4.18) corresponding to the vanishing third-order moments ($p = 0$).

Korneev demonstrated in the paper cited above that the expressions for $l\,(t)$ and $b\,(t)$ derived above remain unaltered when (4.29)- and (4.30)-type hypotheses are applied to homogeneous nonisotropic turbulence.

The following asymptotic formulas for f and h are easily obtained from formulas (4.21) and (4.49) as $\chi = r/l \to \infty$:

$$f = A\left(\frac{l}{r}\right)^{20\alpha} \quad \text{and} \quad h = \frac{2\alpha p}{4\alpha - 1} A\left(\frac{l}{r}\right)^{20\alpha - 1}$$

where

$$A = \frac{2^{30\alpha}\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2} - 10\alpha\right)}$$

and $\Gamma(z)$ is Euler's gamma function. As $r/l \to \infty$, $h \to 0$ only when $\alpha > 0.05$.

The asymptotic formulas for f and h yield that the integration of these functions over r from zero to infinity yields diverging integrals for $\alpha \leqslant 0.1$. The asymptotic formulas obtained above show that

the integral Fourier transforms for the second- and third-order moments, and possibly for the moments of higher order are meaningless. (This remark, as well as the above note on the behaviour of Loitsyansky's invariant, is essential for many theories of turbulence developed on the basis of Fourier transforms.) The corresponding three-dimensional Fourier integrals diverge for $\alpha \leqslant 0.05$.

From the standpoint of physics, the slow decrease of the moments as $r \to \infty$, stemming from the exact solution, is quite acceptable.

In comparing the theory with experiment, it is usual to choose as an example of isotropic turbulence the flow of water or air downstream of stationary two-dimensional gratings with square cells of side length M through which a perturbed flow with the mean translational velocity $\bar{U}$ perpendicular to the middle section of the grating flows. Strictly speaking, this turbulent flow is nonuniform both in the direction of $\bar{U}$ and in the directions normal to $\bar{U}$ owing to the effect of boundary conditions of cross sections of the turbulent flow at the pipe wall or at the boundary layers separating the turbulent flow from the ambient medium. The initial properties of the flow upstream of the grating may also cause inhomogeneity and violate isotropy of the turbulent flow downstream of the grating.

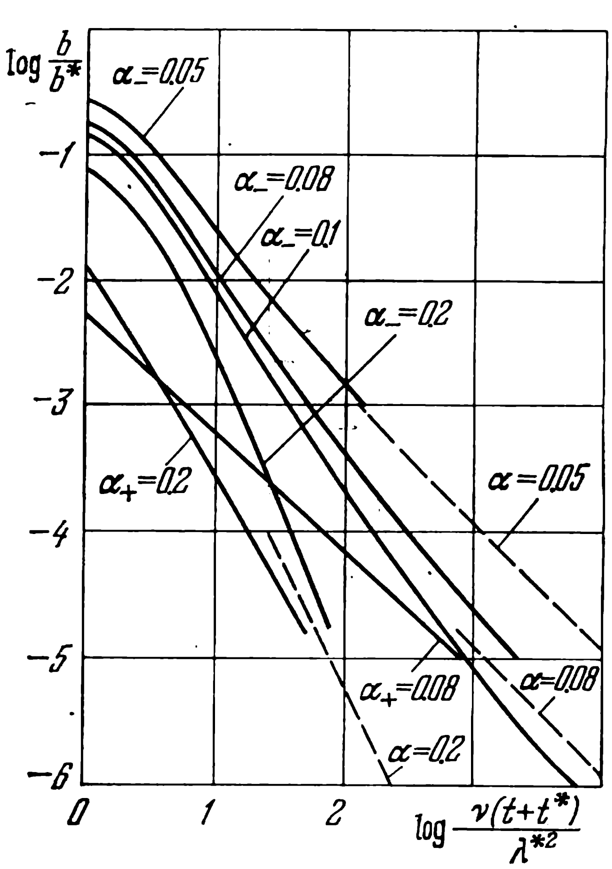

Fig. 25. Theoretical curves of b/b^* as a function of t for different values of α.

It is thus clear that considerable departures from the ideal formulation of the problem, typical for the theory of homogeneous isotropic turbulent motion in an infinite fluid, cannot be neglected in the experimental conditions of the turbulent flow of the fluid downstream of gratings. Naturally, these departures are most pronounced in the immediate vicinity of the grating. Experiment shows that for the Reynolds numbers $\mathrm{Re}_M = UM/\nu \approx 10^3$ the flow can be approximately treated as isotropic at distances from the grating $x > 20M$, but this approximation is valid for correlation coefficients only for not too large r/l because of the phase differences in damping along x; this effect may be of maximum importance for the third- and higher-order moments. Thus, it has been found experimentally [26, 27] that no tendency to isotropic distribution is noticed in

some cases of decay turbulence downstream of gratings. For instance, it has been established in these experiments that the ratio of mean square longitudinal fluctuations to the mean transverse fluctuations of velocity departs from unity, coming to 1.3 ± 0.2.

It must be added to the above arguments concerning the internal properties of turbulent flows that measurements in the experiments to be described below entailed specific errors, and among them the

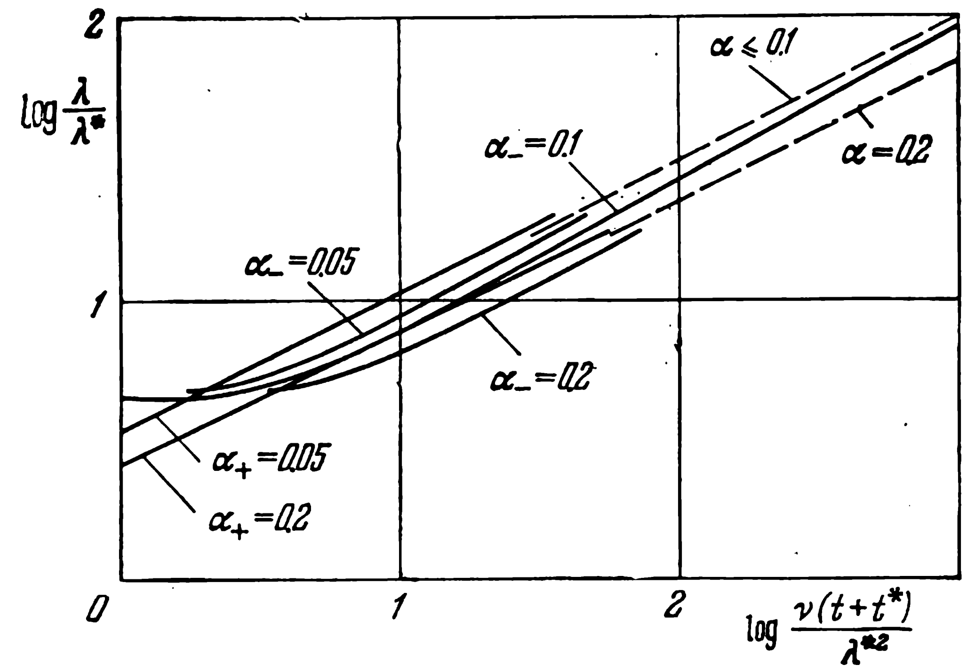

Fig. 26. Theoretical curves of λ/λ^* as a function of t for different values of α.

perturbations due to sensor devices introduced into the flow. Obviously, the listed causes of possible departures of the experimental data from the theory must be borne in mind when the two are compared.

It would seem at the first glance that a qualitative agreement between the theory and experiment is all one needs. However, we will show below that the above-outlined theory leads to a very good quantitative description of the experimental data. This occurs because the initial hypotheses and the derived theoretical relations hold not only for an isolated example, namely, homogeneous isotropic turbulence, but for a number of more general cases [24].

We refer the reader interested in the details of this comparison of the theory and experiment to the already cited paper of Korneev, discussing all experimental data published by 1974. Here we shall give this comparison only with selected experiments, using the appropriate graphs.

We consider the series of experiments of Ling, Huang, and Wan on the decay of the turbulent motion of water downstream of stationary and vibrating gratings [28, 29]. Stationary gratings were arranged with square cells of side length M, made of cylindrical rods of diameter d, with $M/d = 2.8$ and $M = 3.56$ cm in one case (grating A),

$M/d = 2.8$ and $M = 1.78$ cm in the second case (grating B), and $M/d = 5$, $M = 6.4$ cm in the third case (grating C). The vibrating grating was arranged as an array of rods oscillating about their axes, with plates rigidly fixed to the rods (Fig. 27). The maximum velocity V_p of the ends of the plates oscillating together with the rods exceeded the mean translational velocity of water downstream of the grating by a factor of three in one case, and by a factor of 17 in the other. Sufficiently good levelling of mean square components of velocity fluctuations over directions was achieved in these experiments, with the spread not exceeding 5 per cent.

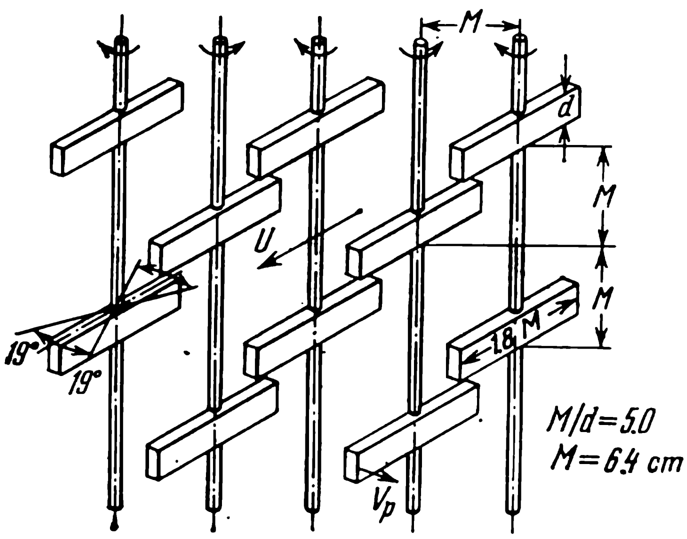

Fig. 27. Schematic view of the grating with vibrating plates used for flow turbulization by Ling and Wan [28, 29] (1972).

Table 1 lists the results of the experiments and the calculated values of the scale factors t^*, b^*, and λ^*. The rms departures of the experimental data for $b(t)$ and $\lambda(t)$ from the theoretical prediction are denoted by ε_b and ε_λ. The theoretical curves plotted by the solid curves in Figs. 28-30 always correspond to the

Table 1

Quantity	Grating type					
	Stationary				Vibrating	
	A	B	C	$A+B$*	$\frac{V_p}{U}=3$	$\frac{V_p}{U}=17$
Re_M	940	470	840	—	2000	2000
M/d	2.8	2.8	5	—	5	5
U, cm/s	2.9	2.9	2.9	2.9	3.14	3.14
M, cm	3.56	1.78	3.18	—	6.4	6.4
t^*, s	−22.2	−17.9	−11.8	−19.7	−5.9	1.25
U^2/b^*	15.9	20	84.7	29.4	33.3	0.445
λ^{*2}/ν, s	14.3	10.9	6.97	17.1	8.12	1.57
ε_b, %	4.5	8	1	3.6	1	2
ε_λ, %	—	9	6	—	2.5	3.8
Notations on the curves in Figs. 28-30	▽	+	×	△	●	○

* Grating A is 30 cm upstream of grating B; the time reference point is chosen at grating A.

same universal value $\alpha_- = 0.08$. An analysis of these graphs indicates a very good agreement between the theory and experiment. Obviously, the relation $b_d^d/b = f\,(r/\lambda)$ is clearly confirmed as revealing self-similarity, and gives a good quantitative fit to theoretical formula (4.48) for $\alpha_- = 0.08$.

The relation $f\,(r/\lambda)$ is also well confirmed by the data on turbulent air flow blown through gratings in the experiments of Batchelor and Townsend [30], although for the Reynolds numbers varying

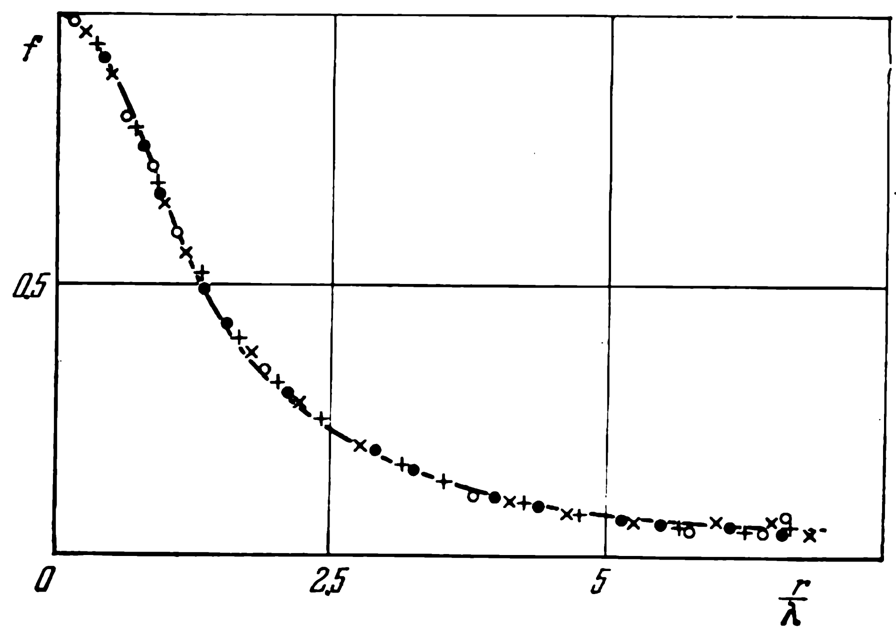

Fig. 28. Comparison of the calculated function $f\,(r/\lambda)$ for $\alpha_- = 0.08$ (solid curve) with the experimental data of Ling, Huang, and Wan [28, 29] (the symbols denoting the experimental data are listed in Table 1).

from $\mathrm{Re}_M = 650$ to $\mathrm{Re}_M = 5620$ the parameter α decreases from $\alpha = 0.2$ to $\alpha = 0.05$, and the corresponding solutions shift from the family α_+ to the family α_-.

Figures 29 and 30 plot the theoretical curves and experimental points for the relations $\frac{b}{b^*}(\tau)$ and $\frac{\lambda}{\lambda^*}(\tau)$ with the experimental results of Ling, Huang, and Wan. The curves again demonstrate a good agreement between the theory and experiment. For large values of t, the curves transform into power function curves (in a logarithmic scale, they transform into linear dependences). If a different suitable value t_0 is used instead of the above-introduced scale factor t^*, the corresponding curves in Figs. 29 and 30 may tend to the respective asymptotic curves more rapidly.

As for the comparison with the experiment of the theoretical functions for the correlation coefficients h and the constant p corresponding to the third-order moments, there can be no serious doubt that there is a good agreement for the homogeneous isotropic turbulence after the above-derived functions $f\,(r/\lambda)$, $\frac{\lambda}{\lambda^*}(\tau)$, and $\frac{b}{b^*}(\tau)$ were found in conformity with the data. Once we are sure that the Kármán-

Howarth equation is a reliable foundation of the theory of homogeneous isotropic turbulence, this equation and the above-mentioned functions for $f(r/\lambda)$, $\frac{\lambda}{\lambda^*}(\tau)$, and $\frac{b}{b^*}(\tau)$ readily yield formulas (4.30) and (4.53) as corollaries. Very scanty experimental data are available in the literature on measuring the third-order moments. A number of researchers simply did not measure the third-order moments; others, Ling and Huang among them, do not publish relevant results for reasons of low accuracy and doubtful reliability. This situation is also caused by a greater sensitivity of the third-and higher-order moments to departures of a turbulent flow from uniformity and isotropy. Only Stewart and Townsend [31] published their results on indirect measurements of $h(r, t)$ in 1951 (no other systematic studies of h have been published since).

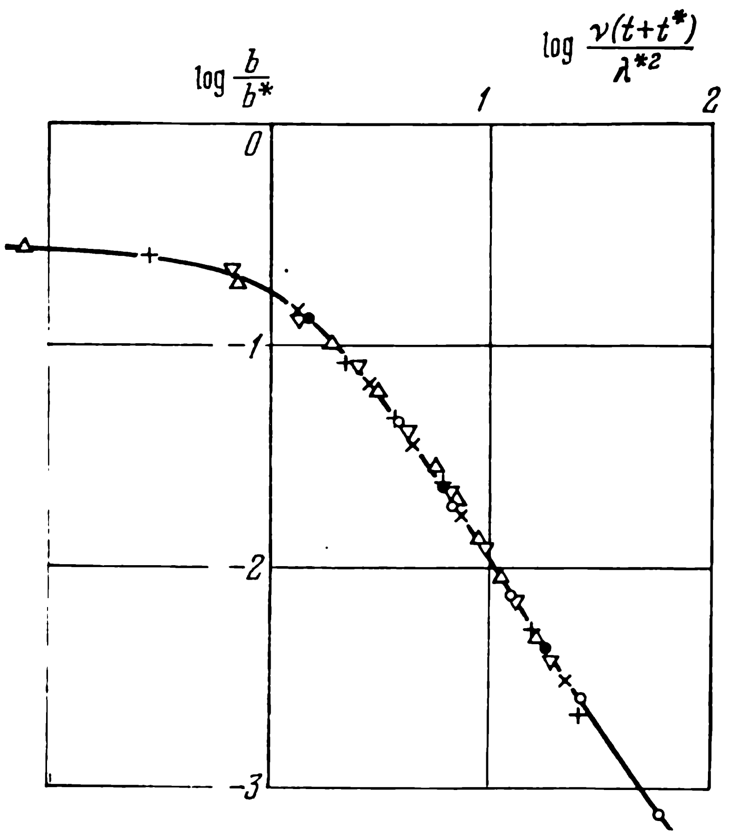

Fig. 29. Comparison of the calculated dependence of b/b^* on time for $\alpha_- = 0.08$ (solid curve) with the experimental data of Ling, Huang, and Wan [28, 29] (the symbols denoting the experimental data are listed in Table 1).

The Stewart and Townsend results are in good qualitative, and for small r good quantitative, agreement with the theoretical universal self-similar relation of the type $p^{-1}h(r/\lambda)$. The spectacular fact is that the corresponding value of λ is found from the experimental data, and the constant $p = \pm\nu/(\lambda^*\sqrt{b^*})$ is also calculated, by virtue of (4.53), from the functions $l(w)$ and $b(t)$. For large r/λ the experimental data of these authors show some, although small, departure from self-similarity. The possible causes of these departures were mentioned earlier in this item.

Virtually no systematic experimental information is available at present on the moments of order higher than three. A theoretical study of higher-order moments can be carried out under an assumption of isotropy for the second- and third-order moments (see the footnote to p. 123), as well as in the framework of a narrower problem of finding isotropy for higher-order moments. Self-similarity hypotheses (4.29) and (4.30) may be referred to the second- and third-order moments only, and then the above-derived equation can be analysed.

When more problems crop up in relation to the moments of order higher than three, too much arbitrariness can be suppressed only by introducing additional hypotheses. Indeed, after the second- and third-order moments are fixed, the successive calculation of higher-order moments can be carried out via the corresponding partial differential equations on the basis of additional conditions. Once the solution for the second- and third-order moments is obtained, one can work out various theories based on different hypotheses in order to describe the turbulent flow of fluids differing only in the behaviour of higher-order moments. Thus, higher-order moments may be treated in terms of the self-similarity assumptions similar to hypotheses [32] (4.29) and (4.30); however, such hypotheses are not mandatory.

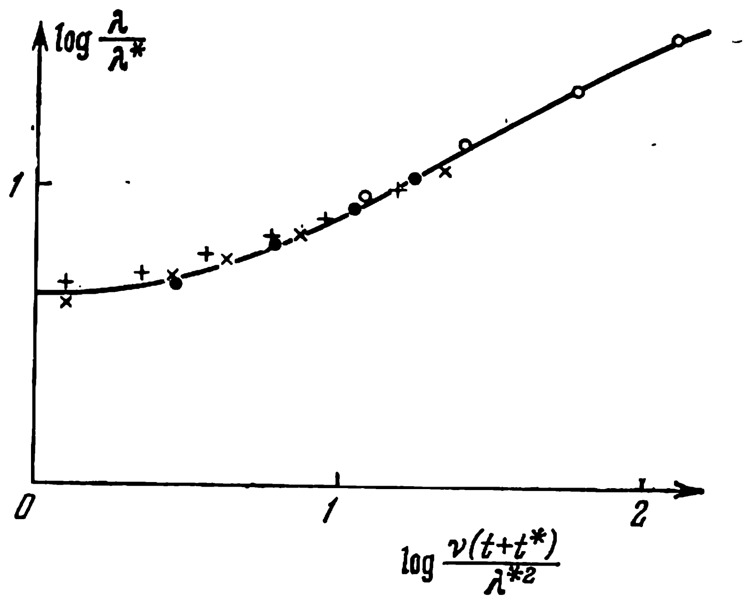

Fig. 30. Comparison of the calculated dependence of λ/λ^* on time for $\alpha_- = 0.08$ (solid curve) with the experimental data of Ling, Huang, and Wan [28, 29] (the symbols denoting the experimental data are listed in Table 1).

§ 5. Steady Turbulent Motion

In a number of cases, in particular, in the fluid motion through pipes and channels, we encounter turbulent motions for which the averaged motion is steady; these are called steady turbulent flows.

We consider the problem of the steady turbulent motion of an incompressible fluid through a fixed smooth circular pipe of infinite length.

We assume that the averaged fluid motion is axisymmetric and the mean velocities are directed parallel to the pipe axis. It follows from the equations of motion of an incompressible fluid that the magnitudes of the mean velocities are independent of the x-coordinate along the pipe axis; the mean velocity is variable over the pipe cross section and depends on the distance r from the centreline of the pipe.

The averaged motion is evidently defined by the following system of parameters:

$$\rho, \ \mu, \ a, \ \tau_0 = -\frac{1}{2}\frac{\partial \bar{p}}{\partial x} a, \quad r = a - y \tag{5.1}$$

where a is the pipe radius, $\partial \bar{p}/\partial x$ is the mean value of the pressure gradient along the pipe, and τ_0 is the friction stress on the pipe walls.

All dimensionless quantities are functions of the two parameters:

$$\mathrm{Re} = \frac{v_* a \rho}{\mu} \quad \text{and} \quad \frac{r}{a} = 1 - \frac{y}{a} \tag{5.2}$$

where $v_* = \sqrt{\tau_0/\rho}$ is called the friction velocity. The dimensionless quantities characterizing the properties of the motion as a whole do not depend on the variable r and, therefore, are defined solely by the Reynolds number.

Let us denote the magnitude of the velocity of the averaged motion by u, and the velocity at the centreline of the pipe by $u_{\max}$. It follows from dimensional analysis that

$$\frac{u_{\max}}{v_*} = f(\mathrm{Re}) \tag{5.3}$$

and

$$\frac{u_{\max} - u}{v_*} = F\left(\mathrm{Re}, \frac{a-y}{a}\right) \tag{5.4}$$

The quantity $u_{\max} - u$ is called the velocity defect, and defines the velocity distribution over the pipe cross section relative to the motion at the centreline.

Later, we shall consider fully developed turbulent motion that corresponds to high values of the Reynolds number.

The velocity distribution in the pipe is closely connected with the turbulent mixing phenomenon; consequently, an exchange of momentum occurs between the adjacent fluid layers. The levelling of velocities as a result of momentum transport is determined by the fluid inertia property.

From the viewpoint of the kinetic theory of matter, the viscosity property is explained by the presence of a random molecular motion which contributes to the levelling of the observed velocities and which leads to the transformation of the kinetic energy of the observed motion into the thermal energy.

The law of conservation of energy states that the sum of the mechanical energy of the observed motion and of the energy of molecular motion is constant. Both kinds of energy can be regarded as components of different kinds of mechanical energy. If the intramolecular forces are neglected, then the viscosity is determined by the mean kinematic characteristics of the state of the molecular motion and by the inertia of the fluid molecules.

The relation between disordered turbulent mixing and the averaged motion is analogous to that between molecular motion and real turbulent motion. Turbulent fluctuations are analogous to the fluctuations of random molecular motion. The difference lies in the different orders of the mean quantities characterizing the fluctuating motion. Instead of the motion of individual molecules in the thermal process during turbulent mixing, we have the fluctuating motion of

"moles", volumes of fluid that are very large in comparison with the size and mass of molecules. Moreover, the magnitudes of the mean fluctuating velocities in turbulent motion are very small in comparison with the magnitude of the mean velocity of thermal motion.

The conversion of the energy of averaged motion into the energy of turbulent molar motion can be analysed in turbulent motion. We can introduce the concept of dissipation of the energy of the averaged motion, where the dissipation is not directly related to the conversion of the mechanical energy into thermal and, therefore, is independent of the fluid viscosity. The redistribution of the kinetic energy between the observed averaged motion and the fluctuating motion can also be considered in an ideal fluid. As is known, the conversion of the mechanical energy into thermal is impossible in an ideal incompressible fluid. Therefore, the conversion of the energy from the averaged motion into the molar turbulent motion of fluctuations can be determined, basically, only by the inertia.

A loss of the mechanical energy occurs during the motion of a fluid in a pipe; therefore, regions must exist in which the effect of viscosity is essential. The instantaneous and mean velocities of the fluid near the wall equal zero because the fluid adheres to the pipe walls. Consequently, intensive fluid mixing cannot exist directly near the pipe wall. This is the basis for the conclusion that the sharp variation in the velocity directly near the wall must be determined by the fluid viscosity and that a laminar layer must exist near a smooth wall. Experimental results are in good agreement with this conclusion.

Let us assume that the levelling of the velocities in the main core of turbulent flow near the pipe axis is determined by the molar mixing of the fluid in which the viscosity has a secondary insignificant value. Let us denote the thickness of a fluid layer near the wall in which the viscosity cannot be neglected by δ. Approximately, δ can be assumed equal to the thickness of the laminar layer near the pipe wall. By our assumption, the viscosity is insignificant for $y > \delta$ and, therefore, the Reynolds number can be omitted in (5.4) for $y > \delta$, i.e.

$$\frac{u_{\max} - u}{v_*} = F\left(\frac{a-y}{a}\right) \tag{5.5}$$

Equation (5.5) indicates the existence of a universal law of velocity distribution in pipes.

Experiments measuring the velocity distribution in pipes during turbulent motion, carried out for all possible values of the Reynolds number, confirm the validity of this universal law, independent of the Reynolds number (Fig. 31). Darcy [33] proposed in 1858 the

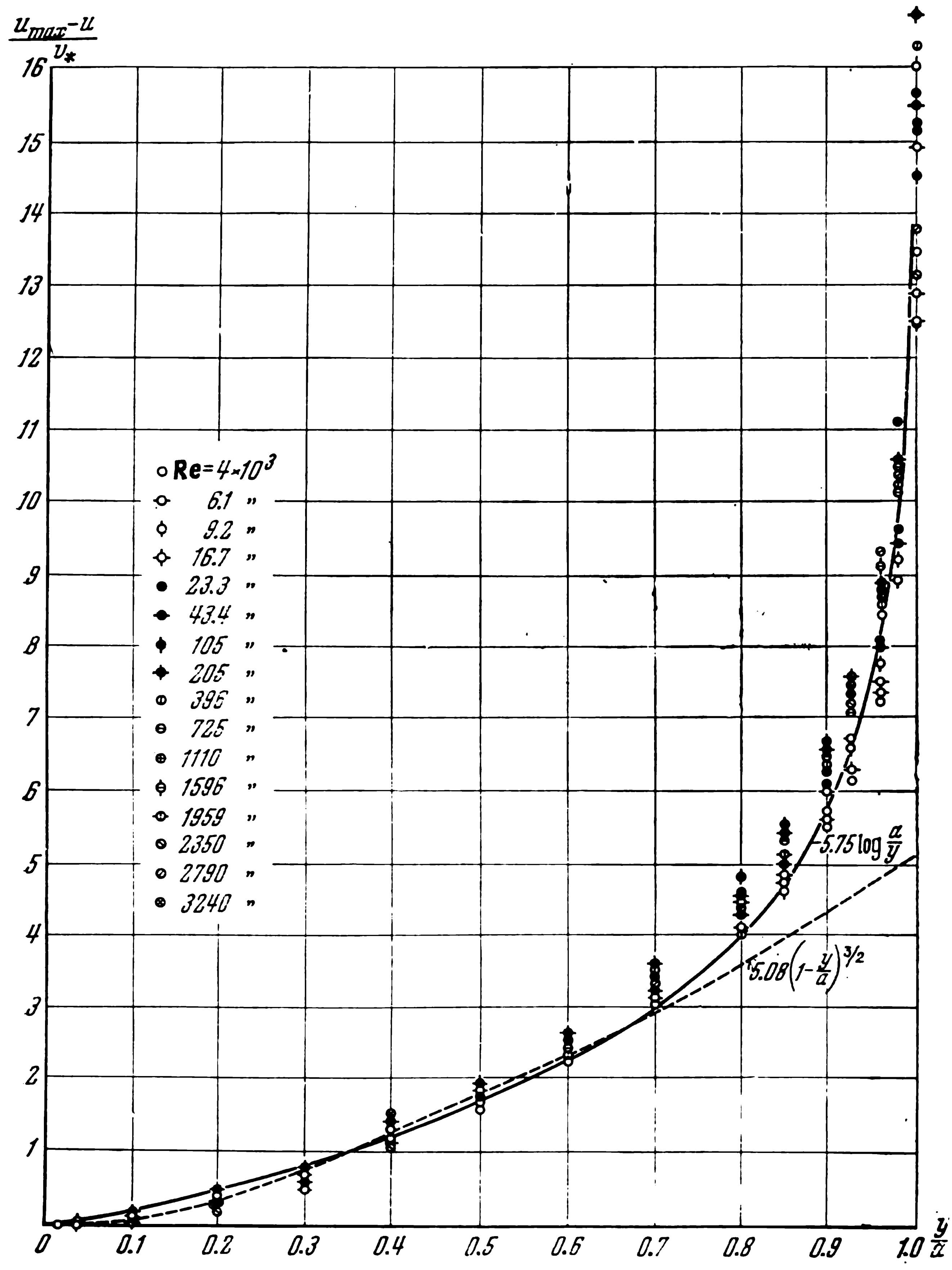

Fig. 31. Experimental confirmation of the universal velocity distribution along a pipe axis.

empirical formula

$$\frac{u_{max}-u}{v_*}=5.08\left(1-\frac{y}{a}\right)^{3/2} \tag{5.6}$$

The graphs taken from experimental results showing $(u_{max}-u)/v_*$ as a function of y/a are also given in [34-36]. These authors noted the existence of the indicated law that appeared to be valid in the centreline of smooth and rough pipes independently of the roughness, even though the resistance, as well as the ratio u_{max}/v_*, etc. depend substantially on the Reynolds number and on the roughness.

This interpretation of the validity of law (5.5) was obtained from dimensional analysis as a consequence of neglecting viscosity in the motion near the pipe centreline. This explanation is given in the works of Prandtl and Kármán.

Now, let us consider the problem of determining the form of the function defining the velocity distribution over the pipe cross section for turbulent motion. Keeping in mind large values of the Reynolds number which are equivalent to large values of the radius a for given v_* and μ/ρ, let us consider the limiting case when $a \to +\infty$. This corresponds to the problem of turbulent motion in the upper half-space $y>0$ bounded by the plane $y=0$.

The system of characteristic parameters will be

$$\rho,\ \mu,\ \tau_0,\ y$$

We obtain the following formula for the mean velocity distribution:

$$\frac{u}{v_*}=\varphi(\eta) \tag{5.7}$$

where $\eta = v_*\rho y/\mu$.

The form of the function $\varphi(\eta)$ for laminar motion is easily determined theoretically. In fact, all the fluid particles move uniformly and in a straight line in laminar motion in a cylindrical pipe; consequently, the inertia is negligible and the velocity distribution does not depend on ρ. Since the characteristic parameters are τ_0, μ, and y, we obtain from dimensional analysis

$$u=k\tau_0\frac{y}{\mu}=k\sqrt{\frac{\tau_0}{\rho}}\,\frac{\sqrt{\frac{\tau_0}{\rho}}\,\rho y}{\mu}$$

where k is a dimensionless constant. From the relation

$$\tau_0=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

we obtain $k=1$. Therefore, for laminar motion, we have

$$\varphi(\eta)=\eta \tag{5.8}$$

If we assume that there is a laminar layer in the general case of turbulent motion near a wall, then (5.8) determines the shape of the function $\varphi(\eta)$ directly at the wall.

The function $\varphi(\eta)$ for turbulent motion can be determined from experiment. But the Reynolds number $\mathrm{Re} = v_* \rho a/\mu$ is involved in addition to the parameter η in experiments in pipes with finite radius.

The empirical power formulas of the type

$$\frac{u}{v_*} = A\eta^n \tag{5.9}$$

where A and n are constant, are used a great deal to determine the velocity distribution in turbulent motion. These constants can be determined either by direct measurement of the velocity distribution or, indirectly, by using an experimental determination of the pipe resistance. In order to clarify the latter method, we express the pipe resistance in terms of the velocity distribution over the pipe radius.

The resistance (drag) coefficient of a circular pipe is determined by

$$\psi = \frac{(p_1 - p_2)\,a}{l \cdot \rho \frac{\bar{u}^2}{2}} = \frac{4\tau_0}{\rho \bar{u}^2} = 4\left(\frac{v_*}{\bar{u}}\right)^2 \tag{5.10}$$

where

$$\bar{u} = \frac{Q}{\pi a^2}$$

$\bar{u}$ is the mean velocity over the pipe cross section, and Q is the volume discharge of the fluid. The formula

$$\frac{u}{v_*} = \varphi\left(\mathrm{Re}_1 = \frac{\rho a \bar{u}}{\mu},\ \frac{\rho v_* y}{\mu}\right)$$

gives the velocity distribution over the pipe radius in the general case. Averaging over the pipe cross section, we obtain

$$\frac{\bar{u}}{v_*} = \frac{1}{\pi a^2}\int_0^a \varphi \cdot 2\pi (a - y)\, dy = 2\int_0^1 \varphi\left(\mathrm{Re}_1,\ \frac{v_*}{\bar{u}}\mathrm{Re}_1 \lambda\right)(1 - \lambda)\, d\lambda \tag{5.11}$$

Solving this equation for $v_*/\bar{u}$, we find ψ as a function of Re_1.

It is not difficult to see that the formula for ψ is

$$\psi = \frac{a}{\mathrm{Re}_1^m} \tag{5.12}$$

where a and m are the constants corresponding to the power law for the velocity distribution defined by (5.9) in which A and n

are independent of the Reynolds number Re_1. In fact, substituting

$$\varphi = A\left(\frac{\rho v_* y}{\mu}\right)^n = A\cdot \mathsf{Re}_1^n \left(\frac{v_*}{\bar{u}}\right)^n \lambda^n$$

in (5.11), we obtain

$$\frac{\bar{u}}{v_*} = \frac{2A}{(n+1)(n+2)} \mathsf{Re}_1^n \left(\frac{v_*}{\bar{u}}\right)^n$$

hence, we determine $v_*/\bar{u}$ and, subsequently, find using (5.10)

$$\psi = 4\left[\frac{(n+1)(n+2)}{2A}\right]^{2/(n+1)} \frac{1}{\mathsf{Re}_1^{2n/(n+1)}} \tag{5.13}$$

Comparing (5.12) and (5.13), we obtain simple relations between the constants m and n and the constants a, A, and n.

The Blasius empirical formula for the resistance of smooth cylindrical pipes is

$$\psi = \frac{0.132}{\mathsf{Re}_1^{1/4}} \tag{5.14}$$

If the power law is used for the velocity distribution, the Blasius formula (5.14) reduces to a "one-seventh" law:

$$\frac{u}{v_*} = 8.7\left(\frac{v_* \rho y}{\mu}\right)^{1/7} \tag{5.15}$$

Formulas (5.14) and (5.15) are in good agreement with experiment for the Reynolds numbers $2\mathsf{Re}_1$ in the 10^4-10^5 range; a formula of type (5.9) with the exponent $n = 1/6$ is in better agreement with experiment for lower values. The exponent must decrease for $2\mathsf{Re}_1 > > 10^5$.

The results of experiments (Fig. 32) show that the best agreement with experiment is obtained by an empirical formula of the form

$$\frac{u}{v_*} = \varphi(\eta) = 5.75 \log \eta + 5.5 \tag{5.16}$$

Formulas (5.9) and (5.16) lose their validity directly at the wall, where $\eta = 0$. There is a laminar layer near the wall for which $\varphi(\eta) = \eta$. If we assume that the laminar layer is adjacent to the turbulent flow and if we require that the velocity of the fluid particles is continuous on transition from the boundary of the laminar layer to the turbulent velocity distribution defined by (5.15) and (5.16), then we have a condition to determine the laminar layer thickness either from the equation

$$\eta = 5.75 \log \eta + 5.5$$

or from

$$\eta = 8.7\eta^{1/7}$$

The solution of these equations yields a value close to $\eta \approx 12$ in both cases. Putting $\eta = 12$, we find the formula for the laminar layer thickness:

$$\frac{\delta}{a} = 12 \frac{\mu}{\rho v_* a} = \frac{24}{\mathrm{Re}_1 \sqrt{\psi}} \tag{5.17}$$

Putting $\mathrm{Re}_1 = 40{,}000$ and using the Blasius formula, we obtain:

$$\frac{\delta}{a} = \frac{68}{\mathrm{Re}_1^{7/8}} = 0.0065$$

Hence, it can be concluded that the laminar layer thickness is small compared with the pipe radius a.

Now, let us consider the theoretical reasoning of Prandtl and Kármán in determining the form of the function $\varphi(\eta)$. We denote

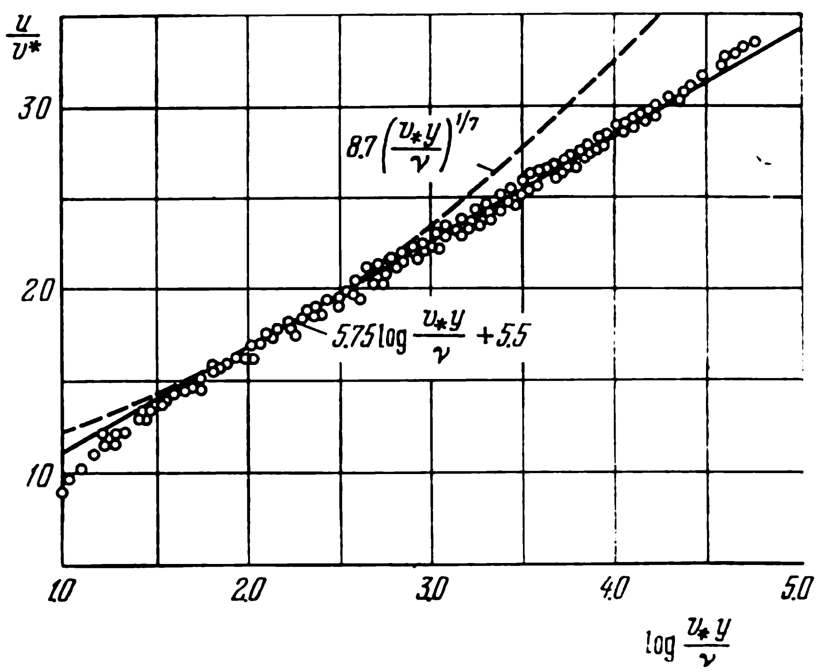

Fig. 32. Velocity distribution in the turbulent boundary **layer.**

the x- and y-components of the turbulent fluctuation velocities by u' and v'. The mean value of the fluid momentum transport along the y-axis, referred to unit time and area, is represented by $\tau = \overline{\rho u'v'}$. Applying the principle of momentum to the fluid volume enclosed within a circular cylinder of radius r coaxial with the pipe (Fig. 33), we obtain after averaging

$$(p_1 - p_2)\,\pi r^2 = \tau \cdot 2\pi r L + \mu \frac{d\bar{u}}{dy}\, 2\pi r L$$

Hence, using the relation

$$(p_1 - p_2)\, a = 2\tau_0 L$$

we find

$$\tau + \mu \frac{d\bar{u}}{dy} = \tau_0 \left(1 - \frac{y}{a}\right) \tag{5.18}$$

For laminar motion $\tau = 0$ and we have the Poiseuille flow. In this case, the parabolic velocity distribution is obtained from (5.18). There is a laminar layer at $y = 0$ in which $\tau = 0$ and $y/a \approx 0$ for turbulent motion near the wall; consequently, (5.18) reduces to

$$\mu \frac{d\bar{u}}{dy} = \tau_0$$

for which

$$\bar{u} = \frac{\tau_0 y}{\mu} \quad \text{or} \quad \frac{\bar{u}}{v_*} = \frac{\rho v_* y}{\mu}$$

which is in agreement with (5.8).

In the kinetic theory of gases, the tangential stress of viscous friction $\mu\,(d\bar{u}/dy)$ can be considered the mean value of the momentum transport per unit time and area, specified by the random thermal motion of individual molecules. In this sense, both terms on the left-hand side of (5.18) are of the same character.

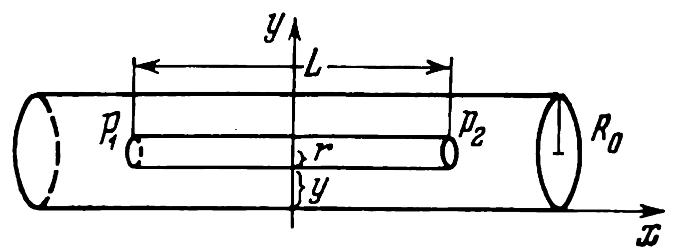

Fig. 33. Diagram for computation of the turbulent motion in a circular cylindrical pipe.

In the region of a fully developed turbulent flow, $\tau \neq 0$ and is large in comparison with $\mu\,(d\bar{u}/dy)$; consequently, it is permissible to neglect the term $\mu\,(d\bar{u}/dy)$ in comparison with τ.

The determination of the variation of τ with the characteristic of the averaged motion can be reduced to the determination of the length l which is defined in terms of τ by the relation

$$\tau = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \tag{5.19}$$

More detailed analysis of the mechanism of turbulent mixing leads to the interpretation of the length l as a quantity analogous to the molecular mean free path in thermal gas motion [37]. Consequently, l is called the mixing length.

The reason for the change from τ to l is related to the physical character of the quantity l. The quantity τ depends on the velocity squared in the absence of viscosity; consequently, l is independent of the velocity, which allows us to use intuitive reasoning to establish the relation with the characteristic dimensions. In a more detailed analysis, it can be shown that l decreases as the wall is approached and l can be related to the roughness characteristics near the wall.

The mixing length for $a = \infty$ is determined by the parameters ρ, μ, v_*, and y; consequently, we have in this case

$$l = y \cdot \Phi\left(\frac{\rho v_* y}{\mu}\right)$$

Let us assume that viscosity is insignificant when the origin of the y-coordinate is suitably chosen. It then follows from this assumption that

$$l = ky$$

where k is a dimensionless constant. Neglecting the terms $\mu\ (d\bar{u}/dy)$ and $y/a \approx 0$ in (5.18), we obtain

$$\tau = \tau_0 = \rho k^2 y^2 \left(\frac{du}{dy} \right)^2$$

Integrating this equation, we find

$$u = \frac{v_*}{k} [\ln y - \ln y_0] \tag{5.20}$$

We have $u = -\infty$ for $y = 0$ on the axis of similarity. The constant of integration y_0 gives the distance between the point at which $u = 0$ and the axis of similarity. A laminar layer, adjacent to the turbulent flow will exist near the wall; if we continue the turbulent flow up to the wall for which the condition $u = 0$ is satisfied, we shall find that y_0 is equal to the distance of the axis of similarity from the wall. Since y_0 must be determined by the quantities ρ, μ, and v_*, we then have

$$y_0 = \beta \frac{\mu}{\rho v_*}$$

where β is a dimensionless constant.

The quantity y_0 is small in fully developed turbulent motion. Substituting the value found for y_0 into (5.20), we find

$$u = \frac{v_*}{k} (\ln \eta - \ln \beta) \tag{5.21}$$

Formula (5.21) can be considered the theoretical basis of empirical formula (5.16) in the region of turbulent motion. The constants k and β must be obtained experimentally. As the Reynolds number increases, the assumptions made in deriving (5.21) become more accurate. This leads to the conclusion that (5.16) is in good agreement with reality when the Reynolds number is sufficiently large.

If it is assumed that the logarithmic velocity distribution (5.21) is correct for turbulent motion in a circular pipe right up to the axis of the pipe, then we obtain

$$\frac{u_{\max} - u}{v_*} = F\left(\frac{y}{a}\right) = 5.75 \log \frac{a}{y}$$

which is in good agreement with experimental data for both smooth and rough-surface pipes; the latter is explained by the fact that the effect of the roughness amounts to a change in y_0, which is excluded in the derivation of this formula [37].

We shall now discuss further applications of dimensional analysis and similarity theory to the problem of turbulent motion.

Let us denote the velocity components of the instantaneous motion by v_x and v_y, and the fluctuation velocity components by u' and v'. In the problem of rectilinear averaged motion, we have

$$v_x = u + u', \quad v_y = v'$$

We consider the velocity field of the relative motion defined by the components $u + u' - u_M$, v', where u_M is the mean velocity at a point M.

The basic Kármán hypothesis is that the turbulent velocity fields of the relative motion are kinematically similar at different points in the flow. Averaging, we find that the field of the mean relative velocities $[u(y) - u_M, 0]$ is also kinematically similar at different points in the flow.

The transformation of the values of all kinematic quantities when we move from one point to another can be expressed in terms of the scales of transformation of two independent kinematic quantities. The magnitudes of these scales can be obtained from an analysis of the mean velocity distribution.

The mean relative velocity distribution near a given point is defined by the successive derivatives of the mean velocity u with respect to y:

$$u', \ u'', \ u''', \ u^{\mathrm{IV}}, \ \ldots \tag{5.22}$$

Any two derivatives have independent dimensions and a dimensionless combination can be formed from any three derivatives. The following dimensionless combination can be formed from the first three derivatives:

$$\frac{u''^2}{u' u'''} = k$$

Since all the derivatives are functions of the same variable, it is evident that a functional relation, which does not contain dimensional constants, holds between any two dimensionless combinations formed from sequence (5.22). For example,

$$\frac{u'''^2}{u'' u^{\mathrm{IV}}} = \Phi\left(\frac{u''^2}{u' u'''}\right)$$

Therefore, a necessary and sufficient condition for the similarity of relative motions for all values of y is:

$$\frac{u''^2}{u' u'''} = k = \mathrm{const} \tag{5.23}$$

Condition (5.23) is a differential equation for $u(y)$. Integrating this equation, we find for $k \neq 1$ and $k \neq 2$:

$$u = A(y+B)^{(2-k)/(1-k)} + C \tag{5.24}$$

where A, B, and C are certain dimensional constants of integration. The solution for $k = 1$ is:

$$u = Me^{y/d} + N \tag{5.25}$$

and, finally, we have for $k = 2$:

$$u = P \ln (y + Q) + S \tag{5.26}$$

where M, d, N, P, Q, and S are constants of integration.

Thus the assumption of exact similarity leads to completely defined special forms of the mean velocity distribution.

Formula (5.26) corresponds to the logarithmic velocity distribution.

The following formulas for the mixing length l and the turbulent momentum transport τ follow from the similarity condition:

$$l = k_1 \frac{u'}{u''} \tag{5.27}$$

and

$$\tau = \rho l^2 u'^2 = \rho k_1^2 \frac{u'^4}{u''^2} \tag{5.28}$$

where k_1 is a dimensionless constant.

For a suitable choice of the origin for y, formula (5.27) for velocity distributions (5.24) and (5.26) leads to $l = k_1 y$ which we obtained earlier as a consequence of the hypothesis that viscosity can be neglected.

The result $\tau = \tau_0 = \text{const}$ corresponds to the velocity distribution (5.26). For the velocity distributions (5.25) and (5.24) τ is obtained as a variable.

Condition (5.23) strongly restricts the possible mean velocity distribution. The original exact condition on similarity can be weakened and the similarity of relative velocity fields in the vicinity of a point can only be assumed to the quantities of the second-order smallness. Only u' and u'' are retained as characteristic quantities in sequence (5.22) under such an assumption and, consequently, (5.23) drops out while formulas (5.27) and (5.28) are retained.

Using (5.28) and (5.18), theoretical formulas can be obtained for the mean velocity distribution [38].

Likudis [39] extended this theory by introducing more complicated functional constraints on the mixing length l; he was then able to take into account the effect of a magnetic field on the turbulent motion of a conducting fluid in a circular cylindrical pipe. The magnetic field was shown to suppress turbulent fluctuations.

A formula for l, taking into account a combined effect of small additives of polymers, the wall roughness, and the Reynolds numbers on the distribution of velocities and drag in turbulent flow through pipes, was derived by Vasetskaya and Ioselevich [40]. These authors have shown that polymer additives increase the thickness of the boundary layer at the wall where the velocity profile changes sharply; this produces a reduction in velocity gradients at the wall and the corresponding drop in friction drag. No boundary layer is formed in the turbulent flow at rough-surface walls and at high Reynolds numbers, so that polymer additives cease to affect mean velocity profiles and friction drag [41].

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CHAPTER IV

ONE-DIMENSIONAL UNSTEADY MOTION OF A GAS

§ 1. Self-Similar Motion of Spherical, Cylindrical, and Plane Waves in a Gas

1. The Concept of Self-Similarity. The motion of a gas or a liquid is said to be one-dimensional when all its properties depend on only one geometric coordinate and on time. It can be shown that the only possible one-dimensional motions are produced by spherical, cylindrical, and plane waves [1]. The methods of dimensional analysis can be used to find exact solutions of certain problems of the one-dimensional unsteady motion of a compressible fluid [2]. Many of these problems are of considerable theoretical and practical interest. But even when the formulation of the problem is not itself of interest, the exact solutions obtained can be used as examples to check the validity of various approximate methods of solving problems of gas dynamics.

To distinguish the problems that can be solved by the methods of dimensional analysis, we analyse the dependent variables and the characteristic parameters of one-dimensional motion. The basic physical variables in the Eulerian approach are the velocity v. the density ρ, and the pressure p, and the characteristic parameters are the linear coordinate r, the time t, and the constants that enter into the equations, the boundary and the initial conditions of the problem. Since the dimensions of the quantities ρ and p contain the symbol of mass, at least one constant a the dimensions of which also contain the symbol of mass must be a characteristic parameter. Without loss of generality, it can be assumed that its dimensions are $[a] = \mathrm{ML}^k\mathrm{T}^s$. We can then write for the velocity, density, and pressure

$$v = \frac{r}{t} V, \quad \rho = \frac{a}{r^{k+3} t^s} \mathcal{R}, \quad p = \frac{a}{r^{k+1} t^{s+2}} P \tag{1.1}$$

where V, $\mathcal{R}$, and P are arbitrary quantities and, therefore, can depend only on dimensionless combinations of r, t, and other parameters of the problem.

In the general case, they are functions of two dimensionless variables. But if an additional characteristic parameter b can be intro-

duced with dimensions independent of those of a, the number of arbitrary variables which can be formed by combining r, t, a, and b is reduced to one. (In general, there may be several characteristic constants but their dimensions must depend on a and b. For what follows it is essential that only two of the characteristic constants, a and b, have independent dimensions and fixed exponents k, s, m, and n which can be integer, or fractional, or transcendental numbers. As for the actual finding of these exponents in specific problems, it is determined by the formulation of the problems and by the properties of the solutions sought, that is, by factors always beyond the scope of dimensional analysis.)

Since the dimensions of the constant a contain the symbol of mass, we can choose the constant b, without loss of generality, so that its dimensions do not contain the symbol of mass, i.e. $[b] = \mathrm{L}^m\mathrm{T}^n$. The single dimensionless independent variable in this case will be

$$\frac{r^m t^n}{b}$$

which can be replaced for $m \neq 0$ by the variable

$$\lambda = \frac{r}{b^{1/m} t^{\delta}} \tag{1.2}$$

where $\delta = -n/m$.

If $m = 0$, then V, $\mathscr{R}$, and P depend only on the time t, the velocity v being proportional to r; the motion corresponding to this special case is studied in detail in § 15.

The solution depending on the independent variable λ may contain a number of arbitrary constants.

This argument shows that, then the characteristic parameters include two constants with independent dimensions in addition to r and t, the partial differential equations, satisfied by the velocity, density, and pressure in the one-dimensional unsteady motion of a compressible fluid, can be replaced by ordinary differential equations for V, $\mathscr{R}$, and P.

The solution of these ordinary differential equations can sometimes be obtained exactly in closed form and, in other cases, approximated by using numerical integration.

Such motions are called self-similar. We now formulate some problems which can easily be solved by the method just described.

To be definite, we assume that the gas is perfect, inviscid, and non-heat-conducting, so that the motion does not involve any kind of physical or chemical change (we shall consider later the extent to which these assumptions are valid in any given problem). The

equations of motion, continuity, and energy take the form

$$\left.\begin{aligned} &\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0\\ &\frac{\partial \rho}{\partial t}+\frac{\partial \rho v}{\partial r}+(\nu-1)\frac{\rho v}{r}=0\\ &\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)+v\frac{\partial}{\partial r}\left(\frac{p}{\rho^{\gamma}}\right)=0 \end{aligned}\right\} \tag{1.3}$$

where γ is the adiabatic exponent; $\nu = 1$ for plane flow, $\nu = 2$ for flow with cylindrical symmetry, and $\nu = 3$ for flow with spherical symmetry.

These equations do not contain any dimensional constants; consequently, the question of the self-similarity of the motion is determined by the number of parameters with independent dimensions introduced by the additional conditions of the problem. If there are only two of these, the motion is self-similar. We shall now consider examples of self-similar problems. (A very large number of papers have been published by now in which analogous self-similar solutions were obtained and analysed for systems of partial differential equations encountered in various branches of science.)

The equations of relativistic gas dynamics contain one fundamental constant: the speed of light in vacuum, c. Consequently, the only dimensionless variable combination for the self-similar one-dimensional motion of a gas in the relativity theory is the variable $\lambda = r/(ct)$.

The class of self-similar one-dimensional solutions in the special relativity theory (SRT) can be defined, therefore, by a single characteristic parameter whose dimensions contain the symbol of mass.

The equations of the general relativity theory (GRT) contain two fundamental constants: the speed of light c and the gravitational constant G. In this case, self-similar solutions, in the observer's reference frame, take form (1.1), where $a = G$, $k = 3$, $s = 2$, and the functions V, $\mathscr{R}$, and P depend only on $\lambda = r/(ct)$.

2. Motion of a Gas with the Given Initial Distributions of Velocity $v_0(r)$, Density $\rho_0(r)$, and Pressure $p_0(r)$ (Cauchy Problem). The general forms of the functions $v_0(r)$, $\rho_0(r)$, and $p_0(r)$ at the initial instant ($t = 0$) are easily determined when the motion that follows for $t > 0$ is self-similar.

In fact, since the whole motion involves only the two constants a and b with the independent dimensions, the initial state must be defined by the three quantities a, b, and r.

Later, we shall assume that the dimensions of b and r are independent and, therefore, that $n \neq 0$. Without loss of generality, let us put $[b] = \mathrm{LT}^{-\delta}$ for $m \neq 0$. (If the dimensions of b and r are related, then, $n = \delta = 0$ and, consequently, a dimensionless combination

br^{-m} exists. It then follows from dimensional considerations that the initial velocity can only equal zero or infinity. The same applies to the initial density and pressure if $s \neq 0$ or $s \neq -2$.

If $n = 0$ and $s = 0$, then the initial density is an arbitrary function of r but the pressure and velocity are either zero or infinite. If $n = 0$ and $s = -2$, then the initial pressure is arbitrary but the velocity and density are either zero or infinite.)

From dimensional considerations, it follows that the initial distributions must be of the form:

$$v_0 = \alpha_1 b^{\frac{1}{\delta}} r^{1-\frac{1}{\delta}}, \quad \rho_0 = \alpha_2 a b^{\frac{s}{\delta}} r^{-\left(k+3+\frac{s}{\delta}\right)}, \quad p_0 = \alpha_3 a b^{\frac{s+2}{\delta}} r^{-\left(k+1+\frac{s+2}{\delta}\right)} \tag{1.4}$$

where α_1, α_2, and α_3 are arbitrary constants; in plane wave motion the values of these constants when $r > 0$ are different from those when $r < 0$.

The uniqueness and existence of the solution of the initial value problem must be considered when the r interval is infinite. The physical characteristics of motion may be infinite when $r = 0$ or $r = \infty$.

To solve the initial value problems for arbitrary values of k, s, and δ, it is necessary to analyse self-similar motions of the most general type.

In the special relativity theory, the initial distributions for velocity, density, and pressure obtained for self-similar one-dimensional solutions have form (1.4), where $\delta = 1$. (The corresponding systems of ordinary equations, their qualitative analysis, and physical interpretation are given in [3-6].) In the general relativity theory, the initial distributions for self-similar solutions must be of form (1.4) but with special fixed values of the constants: $\delta = 1$, $k = -3$, and $s = 2$ [7-11].

3. The Piston Problem. A gas occupies a long cylindrical tube closed at one end by a piston. The gas is at rest ($v_1 = 0$) at the initial instant and the piston starts to move at a constant speed U (Fig. 34).

The characteristic parameters in this problem, in addition to r and t, are the initial density ρ_1, the initial pressure p_1, and the piston velocity U. Since the dimensions of ρ_1, p_1, and U are connected by the relation

$$[U^2] = \frac{[p_1]}{[\rho_1]}$$

just two constants with independent dimensions enter into the problem.

A similar problem can be formulated for motion with cylindrical and spherical symmetry: an infinite region of a gas is set in motion at the initial instant by a cylinder or a sphere, respectively, the

radius of which increases from zero in proportion to the time (Fig. 35).

If the piston velocity is not constant but, for example, is proportional to some power of the time, so that

$$U = ct^n.$$

then the third fundamental constant c appears the dimensions of which (for $n \neq 0$) are independent of the dimensions of ρ_1 and p_1: $[c] = LT^{-n-1}$; consequently, the resulting disturbance of the gas will not be self-similar.

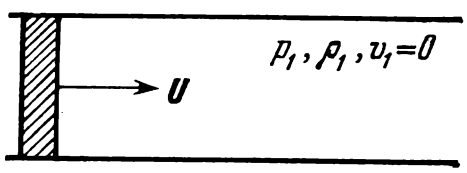

Fig. 34. Piston starts to move at the constant velocity U; the gas ahead of the piston is at rest initially with the uniform density ρ_1 and pressure p_1.

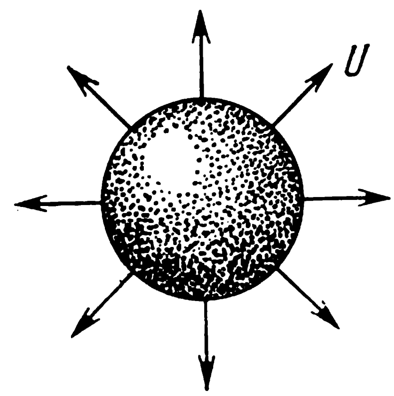

Fig. 35. Expansion of a sphere or a circular cylinder at the constant velocity U. Initially the gas is at rest, and the radius of the sphere or cylinder is zero. The initial gas density ρ_1 and the initial pressure p_1 are constant.

The motion with this variable piston velocity ($n \neq 0$) will be self-similar in the limiting case when $p_1 = 0$. In this case, only the two independent dimensional constants ρ_1 and c enter into the problem.

4. Problem of Gas Converging at and Diverging from a Point. We consider an infinite homogeneous region of a gas in which the pressure and density have constant initial values p_1 and ρ_1 and all particles have the same initial velocity directed either toward the centre (converging) or away from the centre (diverging).

Correspondingly, in the case of cylindrical symmetry, we consider a motion in which all particles have the same initial velocity directed either toward the axis of symmetry or away from it.

Clearly, after adding a translational velocity equal and opposite to the initial velocity of the gas, the analogous problem in plane wave motion reduces to the problem of uniform motion of a piston.

In contrast to the plane flow case, the present problems differ from the problems in the case of spherical and cylindrical symmetry.

As in the preceding problem, there are just the two constants with independent dimensions (p_1 and ρ_1) among the dimensional constants appearing in the initial and boundary conditions.

The problems of converging and of diverging of a gas in a uniform initial state are particular cases of the more general initial Cauchy problem. In other cases, when the characteristic constants have dimensions which are different and independent of the dimensions of velocity, pressure, and density, the initial radial distributions must be variable if they are not zero.

5. Propagation of a Flame Front or Detonation Wave. An infinite region of a homogeneous combustible mixture with the constant density ρ_1 and pressure p_1 is ignited at the time $t = 0$ along a plane (plane case), a line (cylindrical symmetry) or at a point (spherical symmetry). A plane, cylindrical or spherical flame front or detonation wave will then be propagated through the mixture.

As is known, the thickness of the combustion zone is insignificant (of the order of a fraction of one millimetre) under ordinary conditions.

If the processes taking place in the combustion zone itself are not of interest, then we can take its thickness as zero, so that the gas burns instantaneously at a certain geometric surface. In this case, the characteristic parameters will be: the initial density ρ_1 of the mixture, the initial pressure p_1, the quantity of heat Q liberated during the combustion of a unit mass of a gas, and in the case of flame front propagation, its velocity u, which is a constant determined by the physical chemistry of the given mixture.

The dimensions of Q in mechanical units can be expressed in terms of the dimensions of ρ_1 and p_1:

$$[Q] = \frac{[p_1]}{[\rho_1]}$$

Therefore, again only two of the four characteristic parameters have independent dimensions.

If the initial density ρ_1 is variable, obeying the relation

$$\rho_1 = \frac{A}{r^{\omega}}$$

then we have three characteristic constants A, p_1, and Q, and the perturbed gas motion will not be self-similar.

An analysis of the equations of motion and the boundary conditions shows that the initial pressure p_1 only enters into the condition for the shock wave; if the initial pressure p_1 is neglected in this condition in comparison with the large pressure at the detonation wave front, then p_1 is eliminated from the set of characteristic parameters. The problem becomes self-similar in this approximate formulation, with the two independent dimensional constants $[A] = ML^{\omega-3}$ and $[Q] = L^2T^{-2}$.

6. Problem of the Decay of an Arbitrary Discontinuity in a Combustible Mixture. At the time $t = 0$ a gas in a uniform state at the

velocity v_1, density ρ_1, and pressure p_1 occupies the region to the left of the plane $r = 0$ ($\nu = 1$ case). The region to the right of this plane is occupied by a combustible mixture in which the velocity v_2, density ρ_2, and pressure p_2 have constant values. Since, in general, the conditions of conservation of mass, momentum, and energy will not be satisfied for such a discontinuity, it cannot exist in isolation after the initial instant, and a gas motion with one or more surfaces of discontinuity must develop on each of which the conservation conditions will be satisfied (a flame or detonation front may be propagated through the combustible mixture). The parameters of the problem, namely, Q, the quantity of heat liberated during the combustion of a unit mass of a gas, u, the flame front velocity, and v_1, ρ_1, p_1, v_2, ρ_2, and p_2, always include two with independent dimensions. Therefore, the motion which occurs will be self-similar.

To solve the Cauchy problem in the plane case for arbitrary values of the constants k, s, δ, and α_1^+, α_2^+, α_3^+ for $r > 0$, or α_1^-, α_2^-, α_3^- for $r < 0$ in (1.4), we must solve the more general self-similar problem of the decay of the corresponding singularities of the discontinuity at $r = 0$.

When the solution exists and is unique, the decay of the corresponding discontinuity is a local phenomenon determined solely by the character of the singularity.

7. Problem of a Strong Explosion. An explosion occurs at the time $t = 0$ in a gas at rest at the centre of symmetry (a point), so that a finite amount of energy E_0 is liberated instantaneously. In this formulation, we shall neglect the mass and dimensions of the substance liberating the energy. The problem of a strong explosion reflects the essential characteristics of the phenomena in an atomic bomb explosion; we give the relevant experimental data in § 11.

The three constants with independent dimensions enter into the problem: the initial gas density ρ_1, the initial pressure p_1, and the explosion energy E_0.

The system of characteristic parameters influencing the motion of the perturbed gas after the explosion, under adiabatic conditions, is represented by the quantities

$$\rho_1, \ p_1, \ E_0, \ r, \ t, \ \gamma$$

General considerations of dimensional analysis then show that all dependent dimensionless quantities can only depend on the three dimensionless parameters:

$$\gamma, \ \frac{\rho_1^{1/5} r}{E_0^{1/5} t^{2/5}} = \lambda, \quad \frac{p_1^{5/6} t}{E_0^{1/3} \rho_1^{1/2}} = \tau \tag{1.5}$$

of which λ and τ are variables. Experiment and theory show that an abrupt jump in the characteristics of the motion takes place

on the boundaries of the region of perturbed gas motion during an explosion, and a shock wave is formed. This will be a sphere with radius which increases with time. The effect of the initial pressure p_1 and, therefore, of the parameter τ enters only because of the dynamic conditions for the shock wave.

However, if the explosion is strong (E_0 is large), then the pressure behind the shock wave produced by the explosion will be many times larger than the initial pressure in the gas, and the gas motion at small distances from the centre of the explosion will be practically independent of the initial pressure p_1; hence, only the two dimensional constants are essential: ρ_1 and E_0.

Mathematically, neglecting the initial pressure is equivalent to putting the unperturbed pressure $p_1 = 0$ in the shock wave equations; consequently, the parameter p_1 drops out and, therefore, so does the independent variable τ, with the obvious result that the perturbed gas motion can be considered self-similar.

Neglecting the initial pressure, the "counterpressure" p_1, as the shock wave attenuates further is invalid and, therefore, the problem of the perturbed gas motion due to a point explosion at large distances from the centre of the explosion ceases to be self-similar.

We note that it is sufficient to carry out the computation of just one special case in the numerical solution of the problem of a point explosion when the counterpressure p_1 is not neglected; the relation between all the quantities required and the dimensionless quantities λ and τ can then be obtained and, hence, the characteristics of the perturbed explosion field for a given value of γ can easily be obtained for any values of E_0, initial density ρ_1, and initial pressure p_1.

The explosion occurs along a line in the cylindrical symmetry case and along a plane in the plane wave case. Correspondingly, E_0 denotes the energy liberated per unit length or area, respectively. (In certain cases, a strong explosion along a line can be treated as a high-intensity electric discharge in a gas. The dimensions of E_0 vary in the cylindrical and plane cases and the parameters λ and τ vary correspondingly, see §§ 11, 12, 13, and 14.)

Evidently, the explosion problem can be generalized to the case of variable initial density obeying the relation $\rho_1 = A/r^{\omega}$. To do this, the constant A with the dimensions formula $[A] = ML^{\omega-3}$ must be taken as the characteristic parameter instead of ρ_1.

In this case, the following variables can be used instead of the variables of (1.5):

$$\gamma,\ \left(\frac{A}{E_0}\right)^{2/(5-\omega)} \frac{r}{t^{2/(5-\omega)}} = \lambda,\quad \frac{p_1^{(5-\omega)/6}\,t}{A^{1/2}E_0^{(2-\omega)/6}} = \tau$$

The effect of the second parameter can be neglected for small values of the parameter τ because of the high energy E_0 liberated,

the small initial pressure p_1, and the small time interval t, and hence, we obtain a self-similar problem of the propagation of a strong explosion in a medium with a variable density.

The solution of the problems of strong explosions is explained in §§ 11, 12, and 14.

8. Properties of Ideal Media and Self-Similarity. It has been shown above that if the boundary and initial conditions contain only two independent dimensional constants in the determination of one-dimensional unsteady adiabatic motion of an ideal (inviscid) perfect gas, then self-similarity holds. (The adiabatic condition can be replaced by other conditions, for example, the absence of a temperature gradient $\partial T/\partial r = 0$ (infinite heat conduction).)

It is not difficult to see that when the characteristic constants have dimensions dependent on the density ρ_1 and the pressure p_1, the results about the self-similarity of motion apply to other ideal media (those in which tangential stresses are absent, or in which the thermodynamic state is determined by two parameters, for instance, p and ρ).

Actually, the usual reasoning of dimensional analysis shows tha the internal energy per unit mass, which enters into the conditionts for the shock fronts, has the form

$$\varepsilon = \frac{p}{\rho} F\left(\frac{p}{p^*},\ \frac{\rho}{\rho^*},\ \alpha_1,\ \alpha_2, \ldots\right)$$

while entropy is of the form

$$S = AG\left(\frac{p}{p^*},\ \frac{\rho}{\rho^*},\ \beta_1, \beta_2, \ldots\right)$$

where p^* and ρ^* are constants with the dimensions of pressure and density. The dimensional constant A is insignificant since it can be cancelled out in the adiabatic condition; $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ are abstract constants. Evidently, the general form of the equation of state is

$$c_V T = \frac{p}{\rho} H\left(\frac{p}{p^*},\ \frac{\rho}{\rho^*},\ \alpha_1,\ \beta_1, \ldots\right)$$

Hence, it is clear that the addition of the constants p^* and ρ^* to the table of characteristic parameters does not violate self-similarity if the two characteristic dimensional constants have dimensions dependent on p^* and ρ^*.

The internal properties of the medium may be self-similar when certain of the essential functions F, G, and H are independent of the two parameters p/p^* and ρ/ρ^* and depend only on the single

dimensionless parameter $p^{k_1}\rho^{k_2}/C$. In this case, the properties of the medium can be expressed solely in terms of the dimensional constant C in place of the constants p^* and ρ^*. Generally speaking, the dimensional constant C must be introduced as one of the characteristic constants. Self-similar problems are possible for such media and also arise when the dimensions of one of the characteristic constants are fixed and equal to the dimensions of C.

The functions F and H reduce to arbitrary constants in the case of an ideal perfect gas, and the additive constant in the function G is insignificant in many formulations of the problem. Consequently, self-similar motions with two arbitrary independent dimensional constants can be constructed for an ideal perfect gas.

§ 2. Ordinary Differential Equations and the Shock Conditions for Self-Similar Motions

1. Ordinary Differential Equations. In order to solve the problems mentioned, let us derive the equations which V, $\mathcal{R}$, and P must satisfy.

Substituting into (1.3) the expressions for v, ρ, and p in terms of V, $\mathcal{R}$, and P from (1.1) and taking (1.2) into account, we obtain

$$\left.\begin{aligned} &\lambda\left[(\delta-V)V'-\frac{P'}{\mathcal{R}}\right]=V^2-V-(k+1)\frac{P}{\mathcal{R}}\\ &\lambda\left[-V'+(\delta-V)\frac{\mathcal{R}'}{\mathcal{R}}\right]=-s-(k-\nu+3)V\\ &\lambda(\delta-V)\left[\frac{P'}{P}-\gamma\frac{\mathcal{R}'}{\mathcal{R}}\right]\\ &\qquad=-s(1-\gamma)-2-[k(1-\gamma)+1-3\gamma]V \end{aligned}\right\}$$

Introducing a new variable $z=\gamma P/\mathcal{R}$ (the temperature T and the variable z are related by

$$RT=\frac{r^2}{\gamma t^2}z$$

where R is the gas constant), we transform these equations to the following:

$$\frac{dz}{dV}=\frac{z\{[2(V-1)+\nu(\gamma-1)V](V-\delta)^2-(\gamma-1)V(V-1)(V-\delta)\}}{(V-\delta)[V(V-1)(V-\delta)+(\varkappa-\nu V)z]}$$
$$-\frac{[2(V-1)+\varkappa(\gamma-1)]z^2}{(V-\delta)[V(V-1)(V-\delta)+(\varkappa-\nu V)z]} \qquad (2.1)$$

$$\frac{d\ln\lambda}{dV}=\frac{z-(V-\delta)^2}{V(V-1)(V-\delta)+(\varkappa-\nu V)z} \qquad (2.2)$$

$$(V-\delta)\frac{d\ln \mathscr{R}}{d\ln\lambda} = [s+(k-\nu+3)V]-\frac{V(V-1)(V-\delta)+(\varkappa-\nu V)z}{z-(V-\delta)^2} \tag{2.3}$$

where

$$\varkappa=\frac{s+2+\delta(k+1)}{\gamma}$$

(Obviously, the transformation from equations of motion (1.3) to system (2.1)-(2.3) is elementary.)

The dimensions of the constant a can be altered for a given type of self-similar motion by introducing a new constant $a_1 = ab^{\chi}$, the exponent χ being arbitrary. The modified values of k_1 and s_1 are determined by the formulas:

$$k_1 = k+\chi, \quad s_1 = s-\delta\chi$$

The parameter λ can also be changed by introducing a new constant $b_1 = b^m$. The functions $P(\lambda)$ and $\mathscr{R}(\lambda)$ and the parameter λ depend on the choice of the dimensions of the constants a and b; it is evident that the variables z, V, as well as the function $z(V)$, are independent of the choice of the exponents k, s, and m, but are determined completely by the type of self-similar motion, which depends essentially on just the two parameters, $\varkappa$ and δ. When the quantities k and s are replaced by k_1 and s_1 in the expression for $\varkappa$, we obtain $\varkappa = \varkappa_1$.

The marked peculiarity of the function $z(V)$ explains the relation of the field of integral curves of (2.1) in the (z, V) plane to the type of self-similar motion, independently of the method of introducing the characteristic constants a and b.

It is easy to see that the fundamental problem reduces to integrating (2.1). If (2.1) is integrated, then the relation of V and $\mathscr{R}$ to λ is determined from (2.2) and (2.3) by using quadratures.

The plane of the dimensionless variables z and V can be considered for arbitrary non-self-similar motions. A certain curve in the (z, V) plane corresponds to the field of one-dimensional unsteady gas motion at each instant. Points of discontinuity will be represented by shocks (jumps) on this curve. Different curves in the (z, V) plane will correspond to the gas motion at different instants for non-self-similar motions.

The points corresponding to strong discontinuities in the (z, V) plane will move with the passage of time. Different curves correspond to different fixed points in space or to different fixed particles in the (z, V) plane. If the motion is self-similar, then the same integral curve of (2.1) will correspond to the field of the gas motion in the (z, V) plane of different points or particles at different instants. It follows from the formulation of self-similar problems that the

shock coordinate r and the variable $\lambda = r/(bt^\delta)$ at the shock are functions of the time t and of the characteristic dimensional constants a and b. (Sometimes, the gas motion is self-similar but the motion of the boundaries, say, the shock wave, is determined by additional constants and, consequently, the shock coordinate r depends not only on a, b, and t but also on other dimensional constants; the formula λ = const at the shock is not valid in these cases. In conformity with the definitions used, such motions, considered as a whole, will be called non-self-similar although the self-similarity is violated only at the boundary.)

It is impossible to form a dimensionless combination of the three quantities a, b, and t; consequently, the surface of discontinuity is given by

$$\lambda = \lambda_0 = \text{const}, \quad r = \lambda_0 bt^\delta$$

Therefore, fixed values of the variables λ, $\mathscr{R}$, z, P, and V correspond to a shock in self-similar motions. Fixed points correspond to jumps in the (z, V) plane.

The magnitude of the shock velocity c is given by

$$c = \frac{dr}{dt} = \delta \frac{r}{t} \tag{2.4}$$

Evidently, δ is constant for self-similar motions. The velocity of phases in space for $r > 0$ and $t > 0$ is directed away from the centre when $\delta > 0$ and toward the centre when $\delta < 0$. Therefore, the shock wave diverges when $\delta > 0$ and converges when $\delta < 0$, and the velocity of the phase motion is slowed down for $\delta < 0$. If $r > 0$, the time t increases, but $t < 0$, so the motion at the shocks is reverse in character. The particle velocity on the parabola $z = (\delta - V)^2$ equals the speed of sound; this velocity is subsonic above the parabola and supersonic below it. The abstract quantity δ is a certain function of time in the general case of non-self-similar motions.

2. Conditions at Compression Shocks. In the majority of the problems mentioned above, strong discontinuities occur in the flow (shock waves, detonation fronts, flame fronts); consequently, we consider the general relations between V, z, and $\mathscr{R}$ on both sides of the surface of strong discontinuity.

The conditions for conservation of mass, momentum, and energy must be satisfied on crossing a surface of strong discontinuity. Denoting the quantities on one side of the discontinuity by the subscript 1 and on the other side by the subscript 2, we can write:

$$\left.\begin{aligned} \rho_1(v_1 - c) &= \rho_2(v_2 - c) \\ \rho_1(v_1 - c)^2 + p_1 &= \rho_2(v_2 - c)^2 + p_2 \end{aligned}\right\} \tag{2.5}$$

$$\frac{1}{2}(v_1 - c)^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}(v_2 - c)^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} \tag{2.6}$$

These equations are written for a perfect gas, in which it is assumed that the enthalpy i per unit mass of a gas is defined as

$$i = c_p T + \text{const} = \frac{\gamma}{\gamma-1}\,\frac{p}{\rho} + \text{const}$$

We replace the quantities v, ρ, and p in (2.5) and (2.6) by their expressions in terms of V, $\mathscr{R}$, and P from (1.1), the velocity c by $\delta r/t$ according to (2.4), and introduce the variable $z = \gamma P/\mathscr{R}$.

The relations at the shock become

$$\mathscr{R}_1 (V_1 - \delta) = \mathscr{R}_2 (V_2 - \delta)$$

$$V_1 - \delta + \frac{z_1}{\gamma (V_1 - \delta)} = V_2 - \delta + \frac{z_2}{\gamma (V_2 - \delta)}$$

$$(V_1 - \delta)^2 + \frac{2z_1}{\gamma-1} = (V_2 - \delta)^2 + \frac{2z_2}{\gamma-1}$$

Solving these equations for V_2 and z_2 in terms of V_1 and z_1, we find that

$$\left.\begin{aligned} V_2 - \delta &= (V_1 - \delta)\left[1 + \frac{2}{\gamma+1}\,\frac{z_1 - (V_1-\delta)^2}{(V_1-\delta)^2}\right] \\ z_2 &= \left(\frac{\gamma-1}{\gamma+1}\right)^2 \frac{1}{(V_1-\delta)^2} \\ &\quad \times \left[(V_1-\delta)^2 + \frac{2z_1}{\gamma-1}\right]\left[\frac{2\gamma}{\gamma-1}(V_1-\delta)^2 - z\right] \end{aligned}\right\} \qquad (2.7)$$

Knowing the (V_1, z_1) point in the (V, z) plane, we find from (2.7) the (V_2, z_2) point where the curve passes through the shock. We assume that the gas particles pass through the shock from state 1 to state 2. It is clear from the symmetry of (2.5) and (2.6) that the subscripts 1 and 2 in (2.7) can be interchanged.

The points of the parabola

$$z = (V - \delta)^2 \qquad (2.8)$$

transform into themselves. On this parabola, weak discontinuities, i.e. the surfaces of discontinuity of the derivatives, can arise. Actually, equation (2.8), written in terms of dimensional quantities, yields:

$$\frac{\gamma p}{\rho} = (v - c)^2$$

i.e. the square of the particle velocity at the shock equals the square of the speed of sound. Points lying above parabola (2.8) transform into points lying below it, and conversely.

Since z is always positive in the physical sense, then only those cases for which the points of the upper half-plane transform into points of the upper half-plane have physical meaning.

Points on the V-axis, corresponding to the limiting case when $z = 0$, transform into points of the parabola

$$z = \frac{2\gamma}{\gamma - 1} (V - \delta)^2$$

Therefore, transformation (2.7) maps the region between the V-axis and the parabola

$$z = (V - \delta)^2$$

into the region between the parabola

$$z = \frac{2\gamma}{\gamma - 1} (V - \delta)^2$$

and the parabola

$$z = (V - \delta)^2$$

and conversely.

Furthermore, since

$$\frac{\rho_2}{\rho_1} = \frac{\mathscr{R}_2}{\mathscr{R}_1} = \frac{V_1 - \delta}{V_2 - \delta} > 0 \tag{2.9}$$

it is evident that points on different sides of the shock in the (z, V) plane lie on the same side of the line $V = \delta$. Relation (2.9) also follows from (2.7) for any $z_1 \geqslant 0$.

For points above parabola (2.8)

$$z > (V - \delta)^2 \quad \text{or} \quad a^2 = \frac{\gamma p}{\rho} > (v - c)^2$$

and for points below parabola (2.8)

$$z < (V - \delta)^2 \quad \text{or} \quad a^2 = \frac{\gamma p}{\rho} < (v - c)^2$$

In other words, we have: the gas particle velocity relative to the shock is subsonic for $z > (V - \delta)^2$ and supersonic for $z < (V - \delta)^2$. Consequently, the region between the parabola $z = (V - \delta)^2$ and the line $z = 0$ corresponds to the states ahead of compression shocks, and the region between the parabolas $z = 2\gamma/(\gamma - 1)\,(V - \delta)^2$ and $z = (V - \delta)^2$ corresponds to the states behind the compression shocks. (An analysis of the conditions for strong discontinuities in the general case can be found, for example, in [12].)

Points above the parabola

$$z = \frac{2\gamma}{\gamma - 1} (V - \delta)^2$$

transform into points of the lower half-plane according to (2.7) and, therefore, cannot correspond to a state of the gas either ahead of or behind the shock.

The regions in Fig. 36 corresponding to the state of a gas ahead of the shock are shaded vertically and the regions corresponding to the state behind the shock are shaded horizontally.

The direction of the possible transformations from the (V_1, z_1) point to the (V_2, z_2) point are shown by arrows.

We note that if the shock wave is propagated into a gas at rest, i.e. if the (V_1, z_1) point is on the z-axis ($V = 0$), then the (V_2, z_2) point must lie on the parabola

$$z_2 = -\delta (V_2 - \delta) \left(1 + \frac{\gamma - 1}{2\delta} V_2\right) \tag{2.10}$$

This parabola is shown in the lower left part of Fig. 36.

The phase velocities at $\lambda = \text{const}$ (in particular, the shock velocity) $c = \delta r/t$ to the left of the line $V = \delta$ are larger and to the right of this line are less than the gas particle velocity $v = Vr/t$ at the same point in space. The direction of both velocities in space are identical in the first and second cases. The relative particle phase velocity in the first case agrees with the direction of the particle velocity and is opposite to the particle velocity in the second case.

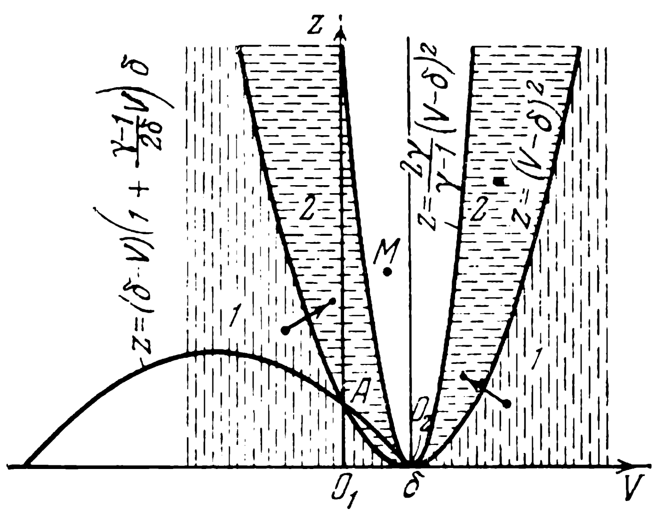

Fig. 36. Transformation of points from the vertically shaded region into the horizontally shaded region corresponds to shocks in the ($z = (\gamma p t^2)/(\rho r^2)$, $V = vt/r$) plane. Possible transformations are shown by arrows.

It is evident that all conclusions of the present item are not related to the self-similarity of motion.

3. Conditions at a Detonation Front and at a Flame Front. Let us first consider the general properties of discontinuities in the case of nonzero energy inflow. Let Q be the energy increment per unit mass when a gas particle passes through a shock front from state 1 to state 2. This flow can be realized by chemical reactions (flame front, detonation front), by heat conduction, heat emission or other appropriate processes.

Mechanical conditions (2.5) still apply when there is the energy increment at the shock front but energy equation (2.6) is changed because of the addition of energy Q to state 2 (an increase of a constant in the enthalpy formula). The modified energy equation will be

$$\frac{\gamma_1}{\gamma_1 - 1} \frac{p_1}{\rho_1} + \frac{1}{2}(v_1 - c)^2 + Q = \frac{\gamma_2}{\gamma_2 - 1} \frac{p_2}{\rho_2} + \frac{1}{2}(v_2 - c)^2 \tag{2.11}$$

In this equation we take account of the fact that the values of the Poisson coefficient $\gamma = c_p/c_V$ ahead of the wave, γ_1, and behind the wave, γ_2, can be different.

Equations (2.5) and (2.11) can be transformed into

$$v_2 - c = \frac{\rho_1}{\rho_2}(v_1 - c) \tag{2.12}$$

$$p_2 = p_1 + \left(1 - \frac{\rho_1}{\rho_2}\right)\rho_1 (v_1 - c)^2 \tag{2.13}$$

$$\left(\frac{\rho_1}{\rho_2}\right)^2 - \frac{2\gamma_2}{\gamma_2+1}\left[1 + \frac{p_1}{\rho_1 (v_1-c)^2}\right]\frac{\rho_1}{\rho_2} + \frac{\gamma_2 - 1}{\gamma_2+1}\left[\frac{2\gamma_1}{\gamma_1-1}\,\frac{p_1}{\rho_1(v_1-c)^2} + 1 + \frac{2Q}{(v_1-c)^2}\right] = 0 \tag{2.14}$$

The absolute value of the difference

$$|c - v_1| = u$$

is the particle velocity relative to the shock ahead of the wave front. The velocity u is a given physico-chemical constant in the case of a flame front.

The following relations are self-evident:

$$\frac{a_1^2}{(v_1-c)^2} = \frac{\gamma_1 p_1}{\rho_1 (v_1-c)^2} = \frac{z_1}{(V_1-\delta)^2}$$

and

$$\frac{a_2^2}{(v_2-c)^2} = \frac{\gamma_2 p_2}{\rho_2 (v_2-c)^2} = \frac{z_2}{(V_2-\delta)^2}$$

In these formulas, z_1 and z_2 are defined by means of γ_1 and γ_2, respectively, and a_1 and a_2 are the corresponding speeds of sound.

The equation

$$\frac{z}{(V-\delta)^2} = \text{const}$$

defines a parabola in the (z, V) plane. The region between the two parabolas in the (z, V) plane

$$z = \alpha (V - \delta)^2 \quad \text{and} \quad z = \beta (V - \delta)^2$$

corresponds to the interval

$$\frac{\gamma p}{\rho (v-c)^2} = \alpha, \qquad \frac{\gamma p}{\rho (v-c)^2} = \beta$$

Equation (2.14) is a quadratic equation defining the ratio ρ_1/ρ_2 as a function of the following four parameters: γ_1, γ_2, $p_1/\rho_1 u^2$, and Q/u^2.

A flame front is a rarefaction (suction) shock for which the following inequalities hold:

$$\frac{\rho_1}{\rho_2} > 1, \quad \frac{\gamma_2 p_2}{\rho_2 (v_2-c)^2} = \frac{z_2}{(V_2-\delta)^2} > 1, \quad \frac{\gamma_1 p_1}{\rho_1 (v_1-c)^2} = \frac{z_1}{(V_1-\delta)^2} > 1 \tag{2.15}$$

i.e. the density of the combustion products is less than the density of the combustible mixture, and the particle shock velocities $(v_1 - c)$ and $(v_2 - c)$ behind and ahead of the flame front are subsonic.

A detonation front is a compression shock for which equation (2.14) has two roots, ρ_1/ρ_{21} and $\rho_1/\rho_{22} \geqslant \rho_1/\rho_{21}$.

The following inequalities hold for these roots:

$$\left.\begin{aligned} &\frac{\rho_1}{\rho_{21}} < 1, \qquad \frac{\gamma_2 p_2}{\rho_2 (v_2 - c)^2} = \frac{z_2}{(V_2 - \delta)^2} \geqslant 1 \\ &\frac{\gamma_1 p_1}{\rho_1 (v_1 - c)^2} = \frac{z_1}{(V_1 - \delta)^2} < 1 \end{aligned}\right\} \tag{2.16}$$

i.e. the fluid particle velocity at the shock is supersonic ahead of the front and subsonic or exactly sonic behind the front;

$$\left.\begin{aligned} &\frac{\rho_1}{\rho_{22}} < 1, \qquad \frac{\gamma_2 p_2}{\rho_2 (v_2 - c)^2} = \frac{z_2}{(V_2 - \delta)^2} \leqslant 1 \\ &\frac{\gamma_1 p_1}{\rho_1 (v_1 - c)^2} = \frac{z_1}{(V_1 - \delta)^2} < 1 \end{aligned}\right\} \tag{2.16$'$}$$

i.e. the fluid particle velocity at the shock is supersonic both ahead of and behind the front. The roots of equation (2.14), ρ_1/ρ_{21} and ρ_1/ρ_{22}, coincide, that is, $\rho_1/\rho_{21} = \rho_1/\rho_{22}$, when the relation

$$\gamma_2^2 \left(1 + \frac{p_1}{\rho_1 u^2}\right)^2 = (\gamma_2^2 - 1) \left[\frac{2\gamma_1}{\gamma_1 - 1} \frac{p_1}{\rho_1 u^2} + 1 + \frac{2Q}{u^2}\right] \tag{2.17}$$

is satisfied.

Since relations (2.12) and (2.13) yield

$$\frac{\gamma_2 p_2}{\rho_2 (v_2 - c)^2} = \gamma_2 \left(1 + \frac{p_1}{\rho_1 u^2}\right) \frac{\rho_2}{\rho_1} - \gamma_2 \tag{2.18}$$

then it is evident from (2.14) that (2.17) is equivalent to the relation

$$\frac{\gamma_2 p_2}{\rho_2 (v_2 - c)^2} = \frac{z_2}{(V_2 - \delta)^2} = 1 \tag{2.19}$$

Therefore, the coincidence of the roots of (2.14) is attained when the particle velocity just behind the shock front is exactly sonic.

Condition (2.17), equivalent to (2.19) for compression shocks, is called the Chapman-Jouguet condition. This condition is realized in actual gas motions involving detonation waves. The front velocity

$$u = v_1 - c$$

can be calculated independently of the nature of a particular problem by using additional equation (2.17) for the given characteristics of the state of gas ρ_1 and p_1 and the given γ_1, γ_2, and Q ahead of the wave front, and the quantites ρ_2, p_2, $v_2 - c$ behind the detonation wave front can be found from (2.12), (2.13), and (2.14).

As early as by the end of the last century, Berthelot, Vielle, Mallard, Le Chatelier, and some others [13] have established that combustion waves are suction shocks, while detonation waves are compression shocks in the combustible medium followed by a very thin zone of a high-rate exothermic reaction. A theoretical treatment of the basic mechanical effects accompanying the steady-state propagation of detonation can already be found in the works of V. A. Michelson (1889), Chapman (1899), Jouguet (1905), Crussard (1907), and others.

In a steady motion of a medium the value of Q is different at different points of the chemical reaction zone, and each state of the intermediate reaction products corresponds to a specific value Q' and moves at the same velocity u with respect to the particles facing the shock front. Obviously, the values v_2', ρ_2', and p_2' corresponding to an intermediate value of Q' are also related via equations (2.12), (2.13), and (2.14). For constant values of ρ_1 and p_1 we obtain, according to (2.13), a straight line in the plane of the variables p_2'/p_1 and ρ_1/ρ_2'; this line is known as the Michelson line [14].

We thus find that all characteristics of the moving medium, v_2', ρ_2', and p_2' can be readily calculated from (2.12), (2.13), and (2.14), and T_2' and S_2' from the equations of state, along the Michelson line for the given initial state p_1, ρ_1, and u in the range from $Q' = 0$ to $Q' = Q$, in terms of Q', p_1, ρ_1, and constant u.

As for determining the same quantities as functions of the coordinate across the chemical reaction zone, it is sufficient to find this for only one of these quantities, using for this purpose the equations of kinetics of the relevant chemical reactions. The corresponding kinetics equations of chemical reactions are often not known in sufficient detail, so that the structure of the detonation wave remains correspondingly undeciphered. Since the chemical reaction layer is thin, it is often possible to restrict the calculations in gasdynamics problems to determining detonation shocks in which the detonation wave is assumed infinitely thin and conditions (2.12), (2.13), and (2.14) at the discontinuity and a given Q are satisfied.

An analysis traceable to Michelson's works shows that in gases

$$\frac{\rho_1}{\rho_2} < \frac{\rho_1}{\rho_{21}} \leqslant \frac{\rho_1}{\rho_{22}}$$

where ρ_2 is the value of the density behind the shock wave front prior to the onset of a chemical reaction at $Q' = 0$, and ρ_1/ρ_{21} and ρ_1/ρ_{22} are the roots of equation (2.14) for a given Q.

The heat release Q' (ρ_1/ρ_2') monotonically increases from zero to Q in the range from ρ_1/ρ_2 to ρ_1/ρ_{21} as the density ρ_2' changes. As ρ_1/ρ_2' continuously increases further along the Michelson line from ρ_1/ρ_{21} to ρ_1/ρ_{22}, Q' varies nonmonotonically, according to (2.12), (2.13), and (2.14): the heat release first continues so that Q' exceeds

Q, then heat is absorbed and Q' drops off and equals Q at ρ_1/ρ_{22}.

This means that if the process in the detonation wave zone involving chemical reactions is continuous, and Q' (ρ_1/ρ_2') can increase only monotonically, then the state behind the detonation front corresponding to the root ρ_1/ρ_{22} cannot be realized. (Obviously, the condition of the monotonicity of the function Q' (ρ_1/ρ_2') is the sufficient condition for the elimination of the root ρ_1/ρ_{22}. On the other hand, the monotonicity of the function Q' (ρ_1/ρ_2') in the chemical reaction zone behind the detonation front is not necessary for the realization of the root ρ_1/ρ_{21}.) The velocity of propagation of detonation through the medium behind the detonation wave front at the point ρ_1/ρ_{21} is either subsonic or is exactly equal to the velocity of sound at the Jouguet point.

Obviously, if the detonation velocity u is strictly subsonic, the detonation wave will interact with the flow of gaseous detonation products, that is, with the gas flow controlled by the boundary conditions behind the detonation wave front. It is this feature, first established in A. A. Grib's thesis in 1939-1940, that clarifies the problem of calculating possible values of the detonation propagation velocity u when the detonation wave propagation is considered in various specific examples.

At small pressure heads behind the shock front, the detonation shock interacting with the flow of detonation products is reduced, and only the minimum detonation front propagation velocity u through the gas particles ahead of the front is possible; this velocity corresponds to the Jouguet point at which the front propagation velocity through the gas particles behind the front is equal to the local sound velocity behind the detonation front. At large pressure heads behind the detonation front (suitable examples are given in § 8) with proper boundary conditions, the detonation front velocity u may be larger than the velocity u found from (2.17) according to the Chapman-Jouguet rule (with the front propagation velocity behind the detonation front through the gas particles being subsonic)

$$(v_2-c)^2 < a_2^2$$

in accordance with the inequality sign in (2.16).

4. On the Relation Between the Temperature and Velocity Behind a Shock. Without loss of generality, let us assume that $v_1=0$, i.e. that the motion relative to the gas ahead of the shock is known.

It then follows from mechanical relations (2.5) that

$$\chi=\frac{v_2^2}{RT_2}=\frac{v_2^2\rho_2}{p_2}=\frac{\left(1-\frac{\rho_1}{\rho_2}\right)^2}{\frac{\rho_1}{\rho_2}\left[\left(1+\frac{p_1}{\rho_1 c^2}\right)-\frac{\rho_1}{\rho_2}\right]} \tag{2.20}$$

The ratio ρ_1/ρ_2 is less than unity for compression shocks. The dimensionless function $\chi(\rho_1/\rho_2, p_1/(\rho_1 c^2))$ is positive and monotone in both the arguments for

$$\frac{p_1}{\rho_1 c^2} > 0 \quad \text{and} \quad \frac{\rho_1}{\rho_2} < 1$$

hence, it follows that in the general case the quantity χ has a maximum value corresponding to $p_1/(\rho_1 c^2) = 0$ and $(\rho_1/\rho_2)_{\min}$:

$$\chi_{\max} = \left(\frac{\rho_2}{\rho_1}\right)_{\max} - 1 \tag{2.21}$$

This value is independent of the energy equation.

Evidently, the greatest compression ρ_2/ρ_1 is attained for simple shock waves or for waves with additional energy absorption. The magnitudes of $(\rho_2/\rho_1)_{\max}$ and $\chi_{\max}$ are less for waves with energy release and, in particular, for flame or detonation waves. It follows from the shock conditions that the greatest value of the ratio ρ_2/ρ_1 is $(\gamma + 1)/(\gamma - 1)$ for simple shock waves and, therefore,

$$\chi_{\max} = \frac{2}{\gamma - 1} \tag{2.22}$$

We find from (2.14), that for strong detonation waves $[p_1/(\rho_1 c^2) = 0]$ satisfying the Chapman-Jouguet condition:

$$\left(\frac{\rho_2}{\rho_1}\right)_{\max} = \frac{\gamma_2 + 1}{\gamma_2}$$

and correspondingly

$$\chi_{\max} = \frac{1}{\gamma_2}$$

The upper limits found for χ permit the temperature behind the front to be found in terms of the fluid particle velocity behind the front.

In may cases, the particle velocity is known; the relative gas velocity for a body moving in a fixed gas equals the body velocity; the particle velocity of a gas in nebulae or in stellar photospheres can be determined in astrophysics by analysing the emission spectra.

For example, if the velocity behind the front in hydrogen is 1000 km/s, then the temperature behind the front in this case must be larger than 5×10^7 °C.

5. Conditions at a Shock Front with Heat Evolution for Self-Similar Motions. The constant Q in this case is the key parameter; since $[Q] = L^2T^{-2}$, we can assume that $m = 2$, $n = -2$, i.e. $\delta = 1$ and, therefore,

$$\lambda = \frac{\beta r}{\sqrt{Q}\, t}$$

where β is a constant whose value can be assigned. We denote the second dimensional constant by A; evidently, we can always take its dimensions formula to be $[A] = ML^{\omega-3}$.

It follows from self-similarity that the density and the pressure at $t = 0$ will be given by

$$\rho_1 = k_1 \frac{A}{r^\omega}, \qquad p_1 = k_2 \frac{AQ}{r^\omega} \tag{2.23}$$

where k_1 and k_2 are constants. If we assume that the gas is initially in equilibrium with no body forces acting, then it follows that $k_2 = 0$ for $\omega \neq 0$ and, therefore, in an unperturbed medium in equilibrium we obtain

$$p_1 = 0 \tag{2.24}$$

If $\omega = 0$, then $p_1 =$ const, and p_1 can be nonzero in equilibrium.

The conditions at the shock become, for self-similar motions with heat evolution,

$$\left.\begin{aligned} \mathscr{R}_1 (V_1 - 1) &= \mathscr{R}_2 (V_2 - 1) \\ V_1 - 1 + \frac{z_1}{\gamma_1 (V_1 - 1)} &= V_2 - 1 + \frac{z_2}{\gamma_2 (V_2 - 1)} \\ \frac{1}{2}(V_1 - 1)^2 + \frac{z_1}{\gamma_1 - 1} + \frac{Q}{c^2} &= \frac{1}{2}(V_2 - 1)^2 + \frac{z_2}{\gamma_2 - 1} \end{aligned}\right\} \tag{2.25}$$

where $\lambda = \lambda^* =$ const at the shock.

For the motion of the shock we have:

$$r_2 = \frac{\lambda^*}{\beta}\sqrt{Q}\, t, \quad c = \frac{dr_2}{dt} = \frac{\lambda^*}{\beta}\sqrt{Q} = \frac{r_2}{t}, \quad \frac{\beta^2}{\lambda^{*2}} = \frac{Q}{c^2}$$

If there are several shocks, then it is always possible to take $\lambda^* = 1$ at one of them. The constant β is, therefore, defined.

The Chapman-Jouguet condition (2.19) yields:

$$z_2 = (V_2 - \delta)^2 \tag{2.26}$$

Hence, the detonation front in the (z, V) plane must correspond to a certain point of parabola (2.26) if the Chapman-Jouguet condition is satisfied.

If the shock is propagated into a gas at rest, then $V_1 = 0$; we obtain from (2.25)

$$\left.\begin{aligned} \mathcal{R}_2 &= \mathcal{R}_1 \left[\frac{\gamma_2}{\gamma_2+1}\left(1+\frac{z_1}{\gamma_1}\right)(1-\Lambda)\right]^{-1} \\ V_2 &= 1-\left[\frac{\gamma_2}{\gamma_2+1}\left(1+\frac{z_1}{\gamma_1}\right)(1-\Lambda)\right] \\ z_2 &= \frac{\gamma_2^2}{(\gamma_2+1)^2}\left(1+\frac{z_1}{\gamma_1}\right)^2(1-\Lambda)(1+\gamma_2\Lambda) \\ &\text{in which} \\ \Lambda^2 &= 1-\frac{(\gamma_2^2-1)\left[\frac{2}{\gamma_1-1}z_1+1+\frac{2Q}{c^2}\right]}{\gamma_2^2\left(1+\frac{z_1}{\gamma_1}\right)^2} \end{aligned}\right\} \tag{2.27}$$

The Chapman-Jouguet condition is equivalent to the condition $\Lambda = 0$; but the Chapman-Jouguet condition is not satisfied, and $V_1 = z_1 = 0$, then

$$\mathcal{R}_2 = \mathcal{R}_1 \frac{\gamma_2+1}{\gamma_2(1-\Lambda)}, \qquad V_2 = \frac{1+\gamma_2\Lambda}{\gamma_2+1},$$

$$z_2 = \frac{\gamma_2^2}{(\gamma_2+1)^2}(1-\Lambda)(1+\gamma_2\Lambda) \tag{2.28}$$

The values of V_2 and z_2 behind the shock are located on the parabola

$$z_2 = \gamma_2 V_2 (1 - V_2) \tag{2.29}$$

The Chapman-Jouguet point corresponds to the point of intersection of parabolas (2.26) and (2.29) at which $\Lambda = 0$. Parabola (2.29) passes through the origin, where $\Lambda = -1/\gamma_2$. The quantity Λ increases as we move upward along parabola (2.29) from the origin, and becomes zero on parabola (2.26); then it increases as we move further along and approaches $\Lambda = +1/\gamma_2$ corresponding to a strong simple shock wave ($Q = 0$).

§ 3. Algebraic Integrals for Self-Similar Motion

The algebraic integrals of the system of ordinary differential equations can be established independently of particular boundary or initial conditions by using dimensional analysis for self-similar motions. In other words, in the general case, the order of the system of ordinary equations can always be lowered.

We shall show below that the number of such integrals can be increased in certain examples when the characteristic constants a and b have particular values. The conclusions which follow remain valid for cases more general than the gas motion described by system (1.3).

To be concrete, we consider the one-dimensional unsteady adiabatic motion of a perfect gas with spherical symmetry taking the Newtonian gravitation into account. We then have the system of equations:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho v}{\partial r}+\frac{(v-1)\,\rho v}{r}=0 \tag{3.1}$$

$$\frac{\partial \mathcal{M}}{\partial r}=\sigma_\nu \rho r^{\nu-1} \tag{3.2}$$

$$\frac{\partial v}{\partial t}+v\,\frac{\partial v}{\partial r}+\frac{1}{\rho}\,\frac{\partial p}{\partial r}+\frac{G\mathcal{M}}{r^{\nu-1}}=0 \tag{3.3}$$

$$\frac{\partial S}{\partial t}+v\,\frac{\partial S}{\partial r}=0 \tag{3.4}$$

where $\sigma_\nu = 2(\nu - 1)\pi + (1/2)(\nu - 2)(\nu - 3)$, G is the gravitational constant, $[G] = M^{-1}L^3T^{-2}$, $\mathcal{M}$ is the mass contained between a fixed surface and the surface under consideration, $[\mathcal{M}] = ML^{\nu-3}$, and S is entropy or a certain function of entropy. In the spherical case, $\nu = 3$. We shall also consider simultaneously the cylindrical wave case $\nu = 2$ (gravitational forces can be taken into account in the following analysis for $\nu = 2$), or the plane wave case $\nu = 1$ when $G = 0$.

Let us consider the self-similar motions defined by the two dimensional constants a and b: $[a] = ML^kT^s$ and $[b] = L^mT^n$. (The expression of the entropy S in terms of p and ρ must not contain essential dimensional constants independent of a and b. It is evident that multiplicative and additive constants are insignificant.) For $m \neq 0$, we can assume without loss of generality that $m = 1$, $n = -\delta$, and $k = -3$. It is sufficient to put

$$a_1 = ab^{(-k-3)/m} \quad \text{and} \quad b_1 = b^{1/m}$$

It is necessary to assume that $[a] = [1/G]$ for $k = -3$ when the Newtonian gravitation is taken into account; consequently, $s = 2$ and, therefore, δ will be the single characteristic parameter. The exponent s can be arbitrary for $G = 0$.

In the general case of self-similar motions when $m \neq 0$, it is possible to write

$$\left.\begin{array}{lll} \lambda = \dfrac{r}{b_1 t^\delta}, & v = \dfrac{r}{t} V(\lambda), & \rho = \dfrac{a}{r^{k+3}t^s}\,\mathcal{R}(\lambda) \\ p = \dfrac{ar^2}{r^{k+3}t^{s+2}} P(\lambda), & \mathcal{M} = \dfrac{ar^\nu}{r^{k+3}t^s} M(\lambda) & \end{array}\right\} \tag{3.5}$$

Substituting (3.5) into (3.1)-(3.4), we obtain a system of four ordinary differential equations for

$$V(\lambda), \quad \mathcal{R}(\lambda), \quad P(\lambda), \text{ and } M(\lambda)$$

We shall now find the algebraic invariants in analytical form relating

$$V,\ \mathcal{R},\ P,\ M,\ \text{and}\ \lambda$$

for this system of equations. There is no need to write these equations down in explicit form.

1. Mass Invariant. It follows from (3.2) that

$$\mathcal{M}'' - \mathcal{M}' = \int_{\mathcal{M}'}^{\mathcal{M}''} d\mathcal{M} = \sigma_\nu \int_{r'}^{r''} \rho r^{\nu-1}\, dr$$

Let us consider the moving surfaces $r'(t)$ and $r''(t)$ for which the parameter λ takes the constant values λ' and λ''. It is not difficult to verify the following identity, which is true for any function $F(r, t)$:

$$\frac{d}{dt}\int_{r'}^{r''} F\sigma_\nu \rho r^{\nu-1}\, dr = \frac{\tilde{d}}{dt}\int_{r'}^{r''} F\sigma_\nu \rho r^{\nu-1}\, dr + \left[F\sigma_\nu\rho r^{\nu-1}\left(\frac{dr}{dt} - v\right)\right]_{r'}^{r''} \qquad (3.6)$$

where the symbol $\tilde{d}/dt$ denotes the time derivative evaluated for a moving volume of integration composed of identical gas particles. Since

$$\mathcal{M}'' - \mathcal{M}' = \frac{ab_1^{\nu-k-3}}{t^{s+\delta(k+3-\nu)}}\left[\lambda''^{\nu-k-3} M(\lambda'') - \lambda'^{\nu-k-3} M(\lambda')\right]$$

then

$$\left.\frac{d(\mathcal{M}'' - \mathcal{M}')}{dt}\right|_{\substack{\lambda'=\text{const}\\ \lambda''=\text{const}}} = -\frac{s+\delta(k+3-\nu)}{t}(\mathcal{M}'' - \mathcal{M}')$$

From the law of conservation of mass, we have

$$\frac{\tilde{d}(\mathcal{M}'' - \mathcal{M}')}{dt} = 0$$

Consequently, (3.6) yields for $F = 1$:

$$-\frac{s+\delta(k+3-\nu)}{t}(\mathcal{M}'' - \mathcal{M}') = \sigma_\nu\left[\rho r^{\nu-1}\left(\frac{dr}{dt} - v\right)\right]_{r'}^{r''}$$

Hence, using (3.5) and the relations $dr'/dt = \delta(r'/t)$ and $dr''/dt = \delta(r''/t)$, we obtain the invariant

$$\lambda^{\nu-k-3}\{[s + \delta(k+3-\nu)]M - \sigma_\nu \mathcal{R}(V - \delta)\} = C = \text{const} \qquad (3.7)$$

which is a corollary of the law of conservation of mass and is, therefore, always true. Invariant (3.7) can be considered independently of the equations of motion in the absence of the Newtonian gravita-

tion as a formula expressing $M(\lambda)$ in terms of λ, V, and $\mathscr{R}$ in finite form. The function $M(\lambda)$ enters into the ordinary differential equation obtained from the momentum equation when gravitation is taken into account. The function $M(\lambda)$ can be eliminated from this equation by using invariant (3.7). If $\mathscr{M} = 0$ or $\mathscr{M} = \text{const}$ for the solution being studied at $\lambda = 0$, i.e. no mass source with a finite or infinite discharge is present at the centre of symmetry, then we obtain from the derivation of invariant (3.7) that, assuming $\lambda' = 0$ and $r' = 0$, the constant C on the right-hand side of (3.7) is zero.

If at $t > 0$ a void (cavity) is formed near the centre of symmetry, and at the boundary of this cavity $\lambda' = \text{const}$ and $\mathscr{M}' = 0$, then the velocity of the particles at this cavity coincides with the cavity expansion velocity $v = dr'/dt$. In this case, we again obtain $C = 0$.

The particular cases discussed above correspond to solutions in which law of mass conservation holds not only at each regular point of the flow at $r \neq 0$ but also at the singular point of the flow at the centre of symmetry for $t \geqslant 0$.

If the law of mass conservation is violated at $r = 0$, then a mass source may be present at the centre of symmetry; in this situation, the constant C can be nonzero.

With no mass sources at the centre of symmetry the variable $\mathscr{M}$ can be treated as a Lagrangian coordinate; otherwise the variable $\tilde{\mathscr{M}} = \rho r^{\nu}(V - \delta)$, having the dimensions of mass, can be taken as the Lagrangian coordinate for $V \neq \delta$. It can be readily verified by resorting to the continuity equation that

$$\frac{d\tilde{\mathscr{M}}}{dt} = 0$$

If $C = 0$, the variables $\mathscr{M}$ and $\tilde{\mathscr{M}}$ differ only in the numerical factor.

The case $V = \delta$ corresponds to a particular solution with the velocity distribution $v = \delta r/t$ linear in radius r; in this case, $\lambda = \text{const}$ for flow particles, and the dimensionless variable λ is therefore a Lagrangian coordinate.

2. Entropy Invariant. In the reversible adiabatic motion of a gas we obtain one integral [15] derived as a corollary of the law of conservation of entropy on particle paths.

Let $\Phi(p, \rho) = f(S)$ be a certain function of entropy. The relation between the entropy S and p and ρ is arbitrary. Condition (3.4) that entropy is conserved on particle paths is equivalent to a relation of the following type:

$$\Phi(p, \rho) = F(\tilde{\mathscr{M}}, a, b, \alpha_1, \alpha_2, \ldots)$$

where $\alpha_1, \alpha_2, \ldots$ are abstract constants, and $\tilde{\mathscr{M}}$ is the Lagrangian coordinate.

We consider the dimensions formula for Φ. Let $[\Phi] = M^{\omega}L^{\mu}T^{\chi}$. If there is no value for any $\varkappa$ for which $[ab_1^{\varkappa}] = [\tilde{\mathcal{M}}]$[1]), then it is impossible to form a dimensionless combination of the three dimensional parameters a, b_1, and $\tilde{\mathcal{M}}$. Consequently, the following equation holds:

$$\Phi(p, \rho) = \tilde{\mathcal{M}}^{\omega}\left(\frac{ab_1^{s/\delta}}{\tilde{\mathcal{M}}}\right)^{\frac{[\mu-\omega(\nu-3)]\delta}{s+\delta(k+3-\nu)}} \times \left(\frac{ab_1^{\nu-k-3}}{\tilde{\mathcal{M}}}\right)^{\frac{\chi}{s+\delta(k+3-\nu)}} f(\alpha_1, \alpha_2, \ldots) \tag{3.8}$$

After p, ρ, and $\tilde{\mathcal{M}}$ have been replaced in (3.8) by (3.5) we obtain the final relation between V, $\mathcal{R}$, P, and λ which is an invariant of the continuity equation and the equation of conservation of entropy.

If an arbitrary combination $ab_1^{\varkappa}/\tilde{\mathcal{M}}$ exists for a certain $\varkappa$, then the quantity f can depend on $ab_1^{\varkappa}/\tilde{\mathcal{M}}$, according to a relation not known in advance; consequently, the entropy equation, in general, cannot be reduced to an invariant.

If the gas is perfect, then we can put $\Phi = p/\rho^{\gamma}$; we have in this case

$$\omega = 1 - \gamma, \quad \mu = 3\gamma - 1, \quad \chi = -2$$

The conservation of the entropy invariant takes the form

$$\frac{P}{\mathcal{R}^{\gamma}} = [\mathcal{R}\lambda^{\nu}(V-\delta)]^{\frac{2-(\gamma-1)s+\delta[k+1-\gamma(k+3)]}{s+\delta(k+3-\nu)}} \times \lambda^{-\frac{[2+\nu(\gamma-1)]s+2(k+3-\nu)}{s+\delta(k+3-\nu)}} f(\alpha_1, \alpha_2, \ldots) \tag{3.9}$$

By means of the invariants derived from the conditions of conservation of mass and entropy, the order of the system of ordinary equations is diminished from four to two.

3. Energy Invariant. We shall show that this invariant exists [16] if a constant with the dimensions $ML^{\nu-1}T^{-2}$, equal to the dimensions of the energy in the spherical case, and to the energy calculated per unit length or area in the cylindrical or plane case, respectively, can be formed from the characteristic constants a and b_1.

---

[1]) Since

$$\tilde{\mathcal{M}} = \frac{ab_1^{\nu-k-3}}{t^{s+\delta(k+3-\nu)}}\lambda^{\nu-k-3}M(\lambda)$$

then the equation $[ab_1^{\varkappa}] = [\tilde{\mathcal{M}}]$ can be satisfied if $s + \delta(k+3-\nu) = 0$ and if $\varkappa = \nu - k - 3$; in this case, invariant (3.7) yields $\lambda^{\nu-k-3}\mathcal{R}(V-\delta) = C$. If $C = 0$, then $V = \delta$ and, therefore, $v = \delta r/t$. We shall study this particular solution in § 15.

We first consider the case when gravitational forces are absent, and ν assumes values 1, 2, and 3.

The total energy between the moving surfaces $r'(t)$ and $r''(t)$ is given by

$$\mathcal{E} = \int_{r'}^{r''} \left(\frac{v^2}{2} + \varepsilon\right) \sigma_\nu \rho r^{\nu-1}\, dr$$

where ε is the internal energy per unit mass. The change in the energy of particles contained between the surfaces $r'' = \text{const}$ and $r' = \text{const}$ at a given instant equals the work of the pressure forces on these surfaces; consequently,

$$\frac{d\mathcal{E}}{dt} = -\sigma_\nu \left(p''v''r''^{\nu-1} - p'v'r'^{\nu-1}\right)$$

Furthermore, it follows from dimensional considerations applied to any self-similar motions ($m \neq 0$) that the quantity $\mathcal{E}$, which has dimensions $ML^{\nu-1}T^{-2}$, is given by the relation

$$\mathcal{E} = ab_1^{\nu-1-k} t^{\delta(\nu-1-k)-2-s} f(\lambda'', \lambda', \alpha_1, \alpha_2, \ldots)$$

where $f(\lambda'', \lambda', \alpha_1, \alpha_2, \ldots)$ is an arbitrary function. Now, let us assume that $r'(t)$ and $r''(t)$ are determined from the conditions $\lambda' = \text{const}$ and $\lambda'' = \text{const}$. Then in the general case:

$$\frac{d\mathcal{E}}{dt} = [\delta(\nu - 1 - k) - 2 - s]\,\frac{\mathcal{E}}{t}$$

Now using these formulas (3.5) and (3.6), and replacing $F(r, t)$ by $v^2/2 + p/(\gamma - 1)\rho$ (to be definite we assume that $\varepsilon = p/(\gamma - 1)\rho$), we easily derive the following relation which holds for any self-similar gas motion:

$$[s + 2 - \delta(\nu - 1 - k)]\, f(\lambda'', \lambda', \alpha_1, \alpha_2, \ldots) = \sigma_\nu \left\{\lambda^{\nu-1-k}\left[PV + (V - \delta)\left(\frac{\mathcal{R}V^2}{2} + \frac{P}{\gamma - 1}\right)\right]\right\}''$$

The relation contains the unknown function $f(\lambda'', \lambda', \alpha_1, \alpha_2, \ldots)$ which is eliminated if

$$s - \delta(\nu - 1 - k) = -2 \tag{3.10}$$

Therefore, if (3.10) holds, we obtain another essential invariant

$$\lambda^{\nu-1-k}\left[PV + (V - \delta)\left(\frac{\mathcal{R}V^2}{2} + \frac{P}{\gamma - 1}\right)\right] = \text{const} \tag{3.11}$$

a corollary of the law of conservation of energy. Noting that we took $m = 1$ in the previous calculations, it is easy to see that (3.10) is equivalent to the relation

$$[\mathcal{E}] = [ab_1^{\nu-1-k}]$$

Therefore, the existence of the energy invariant is equivalent to the condition that the constant $ab_1^{\gamma-1-k}$ has the dimensions of the energy $\mathscr{E}$; the appropriate self-similar motions can be determined by the constant $\mathscr{E}$ and the constant b_1, $[b_1] = LT^{-\delta}$, where the exponent δ can be arbitrary.

This result is not dependent on the use of the relation

$$\varepsilon = \frac{p}{(\gamma-1)\rho}$$

an invariant similar to (3.11) can be written for other forms of the function $\varepsilon(p, \rho)$ if we have the self-similarity of the type described.

Let us consider the case of gas motion with spherical symmetry taking the Newtonian gravitational forces into account.

In motion with spherical symmetry the total energy of gas particles contained in the volume O between two spheres of radii r'' and r' is

$$\mathscr{E} = \int_{r'}^{r''} \left[\frac{\rho v^2}{2} + \rho\varepsilon - \frac{\rho f(\mathscr{M} - \mathscr{M}')}{r}\right] 4\pi r^2\, dr \qquad (3.12)$$

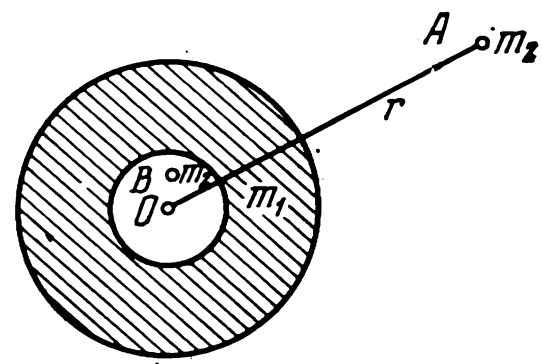

Fig. 37. Spherical layer exerts the same attraction on the external point A as a point mass placed at the centre O. The resultant attraction of the layer is zero for the interior point B.

The first term in the integrand defines the kinetic energy, the second is the thermal (internal) energy, and the third is the part of the internal energy due to the internal gravitational forces. The mass of the gas within the sphere of radius r' is denoted by $\mathscr{M}'$. We can elucidate the third term, which gives the internal energy of the gravitational interaction within the gas. In deriving this formula, the interaction energy is put equal to zero for particles an infinite distance apart. Evidently, the potential energy due to interaction between two point masses m_1 and m_2 equals

$$-\frac{Gm_1m_2}{r_{12}}$$

A spherical layer of a material of total mass $\mathscr{M}_1$, with a density dependent only on the radius (Fig. 37), is known to have the same effect on the external point A with mass m_2 as a point mass $\mathscr{M}_1$ concentrated at the centre of the spherical layer.

The total force of attraction by a spherical layer of the interior point B is exactly zero. Consequently, the potential energy due to attraction of a point mass m_2 located at a distance r from the centre

of the spherical layer and a spherical layer of mass $\mathcal{M}_1$ is

$$-\frac{G\mathcal{M}_1 m_2}{r}$$

The potential energy due to attraction of the point B and a spherical layer is zero.

The mass of the spherical layer between the spheres of radii r and r' equals $\mathcal{M}-\mathcal{M}'$; consequently, the potential energy due to interaction of the mass within the volume O between the spheres r' and r'' is

$$-\int_{r'}^{r''} \frac{G(\mathcal{M}-\mathcal{M}')\,d\mathcal{M}}{r} = -\int_{r'}^{r''} \frac{G(\mathcal{M}-\mathcal{M}')\,\rho 4\pi r^2\,dr}{r}$$

The law of conservation of energy applied to the mass within the volume O gives

$$\frac{\widetilde{d}\mathscr{E}}{dt} = \frac{dA_{\text{sur}}^{(\text{ex})}}{dt} + \frac{dA_{\text{mass}}^{(\text{ex})}}{dt} + \frac{dQ^{(\text{ex})}}{dt}$$

where $dA_{\text{sur}}^{(\text{ex})}/dt$ is the work done per unit time by the external surface forces given by

$$\frac{dA_{\text{sur}}^{(\text{ex})}}{dt} = -p''v''4\pi r''^2 + p'v'4\pi r'^2$$

$dA_{\text{mass}}^{(\text{ex})}/dt$ is the work done per unit time by the external mass gravitational forces which are equivalent to the force of attraction of a point mass $\mathcal{M}'$ at the centre of symmetry. The work of this force on the element $d\mathcal{M}$ per unit time equals:

$$-\frac{G\mathcal{M}'\,d\mathcal{M}}{r^2}\frac{dr}{dt} = G\mathcal{M}'\frac{d}{dt}\left(\frac{d\mathcal{M}}{r}\right)$$

Hence, it follows that

$$\frac{dA_{\text{mass}}^{(\text{ex})}}{dt} = G\mathcal{M}'\frac{\widetilde{d}}{dt}\int_{r'}^{r''}\frac{4\pi r^2\rho\,dr}{r}$$

$dQ^{(\text{ex})}/dt$ is the external heat flow per unit time which equals zero since the process is assumed to be adiabatic.

The law of conservation of energy applied to the gas within the volume O at a given instant then leads to the equation

$$\frac{\widetilde{d}}{dt}\int_{r'}^{r''}\left[\frac{\rho v^2}{2} + \rho\varepsilon - \frac{\rho G\mathcal{M}}{r}\right]4\pi r^2\,dr = 4\pi\left(r'^2p'v' - r''^2p''v''\right) \qquad (3.13)$$

For slow adiabatic gravitational compression of the gas within the spheres ($v = 0$), with no work done by external forces (conditions

that apply to stars), equation (3.13) yields

$$\int_0^R \varepsilon\, d\mathcal{M} = \int_0^R \frac{G\mathcal{M}}{r}\, d\mathcal{M} + \mathrm{const}$$

Since the right-hand side increases with compression, then evidently the gas temperature must rise. The thermal radiation energy is consumed at the expense of the gravitational energy.

If the gas motion is self-similar with the characteristic constants $[a] = \left[\frac{1}{G}\right] = \mathrm{ML^{-3}T^2}$ $(k = -3,\ s = 2)$ and $[b_1] = \mathrm{LT^{-\delta}}$, then we have

$$\mathscr{E}^* = \int_{r'}^{r''} \left(\frac{v^2}{2} + \varepsilon - \frac{G\mathcal{M}}{r}\right) d\mathcal{M} = G^{-1} b_1^5 t^{5\delta-4} f(\lambda', \lambda'') \qquad (3.14)$$

Let us now use (3.6) putting

$$F = \frac{v^2}{2} + \varepsilon - \frac{G\mathcal{M}}{r}$$

Taking (3.13) and (3.14) into account, we find

$$\frac{5\delta-4}{t}\mathscr{E}^* = \left\{4\pi r^2\left[pv + \left(\frac{v^2}{2} + \varepsilon - \frac{G\mathcal{M}}{r}\right)\rho\left(v - \frac{dr}{dt}\right)\right]\right\}_{r'}^{r''}$$

hence, we obtain from (3.5) with $\varepsilon = p/(\gamma - 1)\,\rho$

$$(5\delta - 4) f(\lambda', \lambda'') = \left\{4\pi\lambda^5\left[PV + \left(\frac{V^2}{2} + \frac{P}{(\gamma-1)\mathscr{R}} - M\right)\mathscr{R}(V - \delta)\right]\right\}_{\lambda'}^{\lambda''}$$

The unknown function $f(\lambda', \lambda'')$ drops out of this relation if $\delta = 4/5$. Hence, for $\delta = 4/5$ we get another integral

$$\lambda^5\left[PV + \left(\frac{\mathscr{R}V^2}{2} + \frac{P}{\gamma-1} - \mathscr{R}M\right)\left(V - \frac{4}{5}\right)\right] = \mathrm{const} \qquad (3.15)$$

If $\delta = 4/5$, then

$$\left[\frac{1}{G} b_1^{-5}\right] = \mathrm{ML^2T^{-2}} = [\mathscr{E}^*]$$

Therefore, the gravitational constant G and a particular value of the energy $\mathscr{E}^*$ can be taken as the independent dimensional constants in this case.

4. Momentum Invariant. We consider the case of one-dimensional unsteady motion of plane waves when the constant $c = ab^{\varkappa}$ can be formed from the characteristic constants a and b, for a certain $\varkappa$ having the dimensions of momentum per unit area, $[c] = \mathrm{ML^{-1}T^{-1}}$. In this case, the invariant

$$P - (V - \delta)\mathscr{R}V = \mathrm{const} \qquad (3.16)$$

can be established by arguments similar to those given above and for $k = -1$ and $s = -1$.

The variable λ in any of invariants (3.7), (3.9), (3.11), and (3.15) can be eliminated by a suitable choice of the dimensions index k; k may be varied by replacing the constant a by $a_1 = ab^{\varkappa}$, where $\varkappa$ can assume any arbitrary value.

The previous methods of obtaining invariants are applied to establish the invariants for linearized solutions of nearly self-similar motions [17] and for any approximations [18] in the expansion of non-self-similar solutions in series of self-similar functions.

We have established the finite invariants of the ordinary differential equations for self-similar gas motions. Considerations of dimensional analysis and general theorems of mechanics consistent with the system of equations of motion were used in the derivation. It is clear from the general reasoning that the invariants can also be obtained from the system of ordinary equations by formal calculations.

The finite invariants are derived from a system of ordinary equations which retain the same form for any formulation of the problem; in particular, it holds for the polytropic motion of a perfect gas when $\gamma \neq c_p/c_V$ and, therefore, when the entropy per particle is variable and the external heat flow is nonzero.

§ 4. Motions which Are Self-Similar in the Limit

Starting from a given family of self-similar motions, which depend on several parameters, other families of exact solutions can be constructed by applying certain limit processes to the same governing system of partial differential equations. Let us clarify this by an example.

We take a solution of type (3.5) and write it as follows:

$$\left.\begin{aligned} &v = \frac{r}{t+t_0}\,\delta\widetilde{V}(\lambda), \quad \rho = \frac{a\delta^s\widetilde{\mathscr{R}}(\lambda)}{r^{k+3}(t+t_0)^s} \\ &p = \frac{a\delta^{s+2}}{r^{k+1}(t+t_0)^{s+2}}\,\widetilde{P}(\lambda), \quad \mathscr{M} = \frac{a\delta^s\widetilde{M}(\lambda)}{r^{k+3-\nu}(t+t_0)^s}, \quad \lambda = \frac{r}{b(t+t_0)^{\delta}} \end{aligned}\right\} \tag{4.1}$$

Clearly if the time t is replaced by $t + t_0$ in solution (3.5), we again obtain a solution containing one constant parameter t_0. We now introduce the new notation

$$\left.\begin{aligned} &V = \delta\widetilde{V}, \quad \mathscr{R} = \delta^s\widetilde{\mathscr{R}}, \quad P = \delta^{s+2}\widetilde{P} \\ &M = \delta^s\widetilde{M}, \quad z = \delta^2\widetilde{z} \end{aligned}\right\} \tag{4.2}$$

Equations (3.1), (3.2), (3.3), and (3.4) have a solution of type (4.1) for $k = -3$, $s = 2$ and any values of δ, t_0, and b. For equations

(1.3), $G = 0$, and the constants k and s can be arbitrary. Now, let us put

$$t_0 = \delta\tau \quad \text{and} \quad b = r_0 (\delta\tau)^{-\delta}$$

Evidently, $[\tau] = [t]$ and $[r_0] = [r]$, where τ, r_0, and δ are arbitrary constants.

Proceeding to the limit in equations (4.1), when $\delta \to \infty$ for fixed τ, r_0 and finite $\tilde{V}$, $\tilde{\mathscr{R}}$, $\tilde{P}$, and $\tilde{M}$, we obtain

$$\left.\begin{aligned} &v = \frac{r}{\tau} \tilde{V}(\lambda), \quad \rho = \frac{a}{r^{k+3}\tau^s} \tilde{\mathscr{R}}(\lambda) \\ &p = \frac{a}{r^{k+1}\tau^{s+2}} \tilde{P}(\lambda), \quad \mathscr{M} = \frac{a}{r^{k+3-\nu}\tau^s} \tilde{M}(\lambda), \quad \lambda = \frac{r}{r_0} e^{-t/\tau} \end{aligned}\right\} \tag{4.3}$$

It follows that the equations of one-dimensional unsteady gas motion have solutions of type (4.3), which can be considered to define the limiting forms of self-similar motions. Equations for $\tilde{z}$, $\tilde{\mathscr{R}}$, $\tilde{P}$, and λ can easily be obtained from (2.1), (2.2), and (2.3); we find for finite k and s

$$\left.\begin{aligned} \frac{d\tilde{z}}{d\tilde{V}} = {} & \frac{\tilde{z}\{[2+\nu(\gamma-1)]\tilde{V}(\tilde{V}-1)^2 - (\gamma-1)\tilde{V}^2(\tilde{V}-1)\}}{(\tilde{V}-1)\left[\tilde{V}^2(\tilde{V}-1) + \left(\frac{k+1}{\gamma} - \nu\tilde{V}\right)\tilde{z}\right]} \\ & - \frac{\left[2\tilde{V} + \frac{k+1}{\gamma}(\gamma-1)\right]\tilde{z}^2}{(\tilde{V}-1)\left[\tilde{V}^2(\tilde{V}-1) + \left(\frac{k+1}{\gamma} - \nu\tilde{V}\right)\tilde{z}\right]} \\ \frac{d\ln\lambda}{d\tilde{V}} = {} & \frac{\tilde{z} - (\tilde{V}-1)^2}{\tilde{V}^2(\tilde{V}-1) + \left(\frac{k+1}{\gamma} - \nu\tilde{V}\right)\tilde{z}} \\ (\tilde{V}-1)\frac{d\ln\tilde{\mathscr{R}}}{d\ln\lambda} = {} & -\frac{\tilde{V}^2(\tilde{V}-1) + \left(\frac{k+1}{\gamma} - \nu\tilde{V}\right)\tilde{z}}{\tilde{z} - (\tilde{V}-1)^2} + (k+3-\nu)\tilde{V} \end{aligned}\right\} \tag{4.4}$$

The mass and entropy invariants for these limiting motions become

$$\left.\begin{aligned} &\lambda^{\nu-3-k}\{\tilde{M}(k+3-\nu) - \sigma_\nu \tilde{\mathscr{R}}(\tilde{V}-1)\} = \text{const} \\ &\frac{\tilde{z}}{\tilde{\mathscr{R}}^{\gamma-1}} = \tilde{M}^{-\frac{(k+3)(\gamma-1)+2}{k+3-\nu}} \text{const} \end{aligned}\right\} \tag{4.5}$$

The energy invariant is

$$\tilde{P}\tilde{V} + (\tilde{V}-1)\left(\frac{\tilde{\mathscr{R}}\tilde{V}^2}{2} + \frac{\tilde{P}}{\gamma-1}\right) = \text{const} \tag{4.6}$$

for $k = \nu - 1$ and $s = -2$.

It can easily be shown that the case $s = s_0\delta + s_1$, i.e. $s \to +\infty$ as $\delta \to \infty$, reduces to the case considered if the constant a/τ^s is

replaced by a_1 in equation (4.3), and k by $k+s_0$ in equations (4.4) and invariants (4.5).

Similar considerations apply to the coordinate r which can be replaced by $x+x_0$ for plane wave motions of a gas.

Formulas (3.5) can be written as

$$\left.\begin{array}{ll} v=\dfrac{(x+x_0)\,\delta}{t}\,\hat{V}(\hat{\lambda}), & \rho=\dfrac{a\hat{\mathscr{R}}(\hat{\lambda})}{[(x+x_0)\,\delta]^{k+3}\,t^{s}} \\ p=\dfrac{a\hat{P}(\hat{\lambda})}{[(x+x_0)\,\delta]^{k+1}\,t^{s+2}}, & \mathscr{M}=\dfrac{a\hat{M}(\hat{\lambda})}{[(x+x_0)\,\delta]^{k+2}\,t^{s}} \\ \multicolumn{2}{c}{\hat{\lambda}=\lambda^{1/\delta}=\left(\dfrac{x+x_0}{b}\right)^{1/\delta}\dfrac{1}{t}} \end{array}\right\} \tag{4.7}$$

Here we used the notation

$$V=\delta\hat{V},\quad \mathscr{R}=\frac{1}{\delta^{k+3}}\hat{\mathscr{R}},\quad P=\frac{1}{\delta^{k+1}}\hat{P},\quad M=\frac{\hat{M}}{\delta^{k+2}} \tag{4.8}$$

$$(z=\hat{z}\delta^2)$$

Let us put

$$x_0=\frac{r_0}{\delta},\quad (b)^{1/\delta}=\frac{1}{\tau}\left(\frac{r_0}{\delta}\right)^{1/\delta}$$

We have the following dimensional relations:

$$[x]=[r_0]\quad\text{and}\quad [t]=[\tau]$$

where τ and r_0 can be arbitrary. If we fix k, s, r_0, and τ and let δ tend to zero, we obtain

$$\left.\begin{array}{ll} v=\dfrac{r_0}{t}\hat{V}(\hat{\lambda}), & \rho=\dfrac{a}{r_0^{k+3}t^{s}}\hat{\mathscr{R}}(\hat{\lambda}) \\ p=\dfrac{a\hat{P}(\hat{\lambda})}{r_0^{k+1}t^{s+2}}, & \hat{\lambda}=\dfrac{\tau}{t}e^{x/r_0} \end{array}\right\} \tag{4.9}$$

It is easy to obtain equations for $\hat{z}$, $\hat{P}$, $\hat{\mathscr{R}}$, and $\hat{\lambda}$ from (2.1), (2.2), and (2.3); after substituting (4.8) into (2.1), (2.2), and (2.3), and proceeding to the limit at $\delta=0$, we obtain

$$\left.\begin{array}{l} \dfrac{d\hat{z}}{d\hat{V}}=\dfrac{\hat{z}\left[2(\hat{V}-1)^2-(\gamma-1)\hat{V}(\hat{V}-1)+\left(s-\dfrac{s+2}{\gamma}\right)\hat{z}\right]}{(\hat{V}-1)\left[\hat{V}(\hat{V}-1)-\dfrac{s+2}{\gamma}\hat{z}\right]} \\ \dfrac{d\ln\hat{\lambda}}{d\hat{V}}=\dfrac{\hat{z}-(\hat{V}-1)^2}{\dfrac{s+2}{\gamma}\hat{z}-\hat{V}(\hat{V}-1)} \\ (\hat{V}-1)\dfrac{d\ln\hat{\mathscr{R}}}{d\ln\hat{\lambda}}=s-\dfrac{\dfrac{s+2}{\gamma}\hat{z}-\hat{V}(\hat{V}-1)}{\hat{z}-(\hat{V}-1)^2} \end{array}\right\} \tag{4.10}$$

In this case, the mass and entropy invariants can be written as:

$$\left.\begin{aligned} s\hat{M} - \hat{\mathcal{R}}(\hat{V}-1) &= \text{const} \\ \frac{\hat{z}}{\hat{\mathcal{R}}^{\gamma-1}} &= \hat{M}^{(1-\gamma)+(2/s)}\,\text{const} \end{aligned}\right\} \qquad (4.11)$$

The following energy invariant holds for $k = 0$ and $s = -2$:

$$\hat{P}\hat{V} + (\hat{V}-1)\left(\frac{\hat{\mathcal{R}}\hat{V}^2}{2} + \frac{\hat{P}}{\gamma-1}\right) = \text{const} \qquad (4.12)$$

(The equations of motion can be integrated in quadratures if use is made of invariants (4.5) and (4.6) in addition to (4.11) and (4.12). N. N. Kochina performed this integration [19].)

Proceeding to the limit once more as $\mu \to +\infty$ in (4.3) and (4.9), we obtain general formulas for the steady gas motion after replacing t by $t + \mu t_1$ and τ by μt_1. If we let $\mu \to +\infty$ in (4.9) after replacing x by $x + \mu l$ and r_0 by μl, we obtain a translational motion in which the pressure varies with time.

K. P. Stanyukovich [20] analysed solutions of types (4.3) and (4.9) by formal substitution.

§ 5. Investigation of the Family of Integral Curves in the (z, V) Plane

To analyse the existence, uniqueness, and construction of solutions of different boundary- and initial-value problems, it is necessary to investigate the family of integral curves in the (z, V) plane. When gravitational forces are absent, we consider ordinary differential equation (2.1) which contains the two characteristic parameters $\varkappa$ and δ in addition to the parameter $\nu = 1$, 2, and 3.

The following two families of solutions whose characteristic parameters have the dimensions described below are especially important in the problems discussed above in § 1 and in later applications.

1. One of the constants, u, has the dimensions of velocity; the arbitrary dimensions of the second constant, A, can be taken equal to $\mathrm{ML}^{\omega-3}$ ($k = \omega - 3$ and $s = 0$).
2. One of the constants, E, has the dimensions of energy $\mathrm{ML}^{\nu-1}\mathrm{T}^{-2}$; the dimensions of the second constant, A, can either be arbitrary or taken equal to $\mathrm{ML}^{\omega-3}$.

Case 1 covers the problems in which the phase velocities are constant: the motion of a piston with constant speed in a medium with constant initial pressure and density ($\omega = 0$), the detonation and combustion in a medium with constant density or density varying as $\rho_1 = A/r^{\omega}$, and the decay of an arbitrary discontinuity in a combus-

tible mixture with uniform conditions in the gas ahead of and behind the front.

Case 2 covers the problems of a strong explosion ($p_1 = 0$) with constant initial density when $\omega = 0$, or with variable initial density $\rho_1 = A/r^{\omega}$ when $\omega \neq 0$.

We have in case 1

$$\delta = 1, \quad \varkappa = \frac{\omega}{\gamma}, \quad \lambda = \beta \frac{r}{ut} \tag{5.1}$$

and in case 2

$$\left.\begin{aligned} & b = \left(\frac{E}{A}\right)^{1/(2+\nu-\omega)}, \quad [b] = \mathrm{LT}^{-2/(2+\nu-\omega)}, \quad \delta = \frac{2}{2+\nu-\omega} \\ & \varkappa = \frac{\nu\delta}{\gamma}, \quad \lambda = \beta \frac{r}{bt^{\delta}} \end{aligned}\right\} \tag{5.2}$$

where β is an arbitrary abstract constant which must be chosen suitably in each specific solution. In both cases, the families of solutions depending only on the single characteristic parameter ω are obtained in the (z, V) plane.

We now consider differential equations (2.1) and (2.2) in detail.

We have in case 1:

$$\frac{dz}{dV} = \frac{z\left\{2(V-1)^3 + (\nu-1)(\gamma-1)V(V-1)^2 - \left[2(V-1) + \omega\frac{\gamma-1}{\gamma}\right]z\right\}}{(V-1)\left[V(V-1)^2 + \left(\frac{\omega}{\gamma} - \nu V\right)z\right]} \tag{5.3}$$

$$\frac{d\ln\lambda}{dV} = \frac{z-(V-1)^2}{V(V-1)^2 + \left(\frac{\omega}{\gamma} - \nu V\right)z} \tag{5.4}$$

Equation (2.3) can be replaced by the entropy invariant. From (3.9), for $\delta = 1$, $s = 0$, $k = \omega - 3$, and using (3.7), we obtain

$$\frac{z}{\mathscr{R}^{\gamma-1}} = C_1\left[\mathscr{R}(V-1) + \frac{C_2}{\lambda^{\nu-\omega}}\right]^{\frac{\omega(\gamma-1)}{\nu-\omega}} \frac{1}{\lambda^2} \tag{5.5}$$

When (5.3) is integrated and $\lambda(V)$ is calculated from (5.4) by quadrature, (5.5) determines the function $\mathscr{R}(V)$; C_1 and C_2 are arbitrary constants. If the mass $\mathscr{M}$ equals zero for $\lambda = r/r_2 = 0$ and there is no mass source, then

$$C_2 = 0$$

Ordinary differential equation (5.3) in the half-plane $z \geqslant 0$ has the following singular points in the spherical case for $\nu = 3$ (see Figs. 38-41).

The point O ($z = 0$, $V = 0$) is a node; the integral curves touch the V-axis at the origin O, but there is the only curve that approaches the point O with the slope γ/ω. The following asymptotic

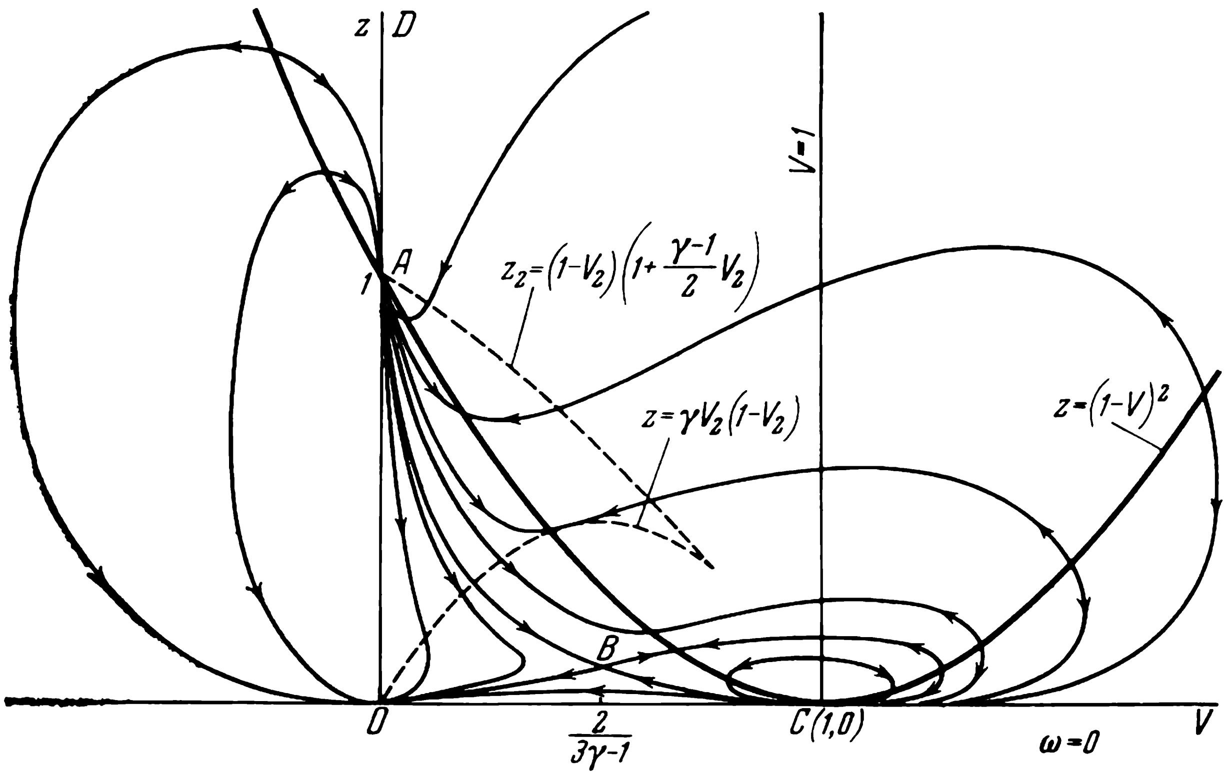

Fig. 38. The family of integral curves for $\nu = 3$, $\delta = 1$, and $\omega = 0$. Points corresponding to the state of rest on the line OA transform after passing through the shock wave into points of the parabola $z_2 = (1-V_2)\left(1 + \frac{\gamma - 1}{2} V_2\right)$. Arrows indicate the direction of increasing $\lambda = \beta r/(ut)$. For $V > 0$, $\lambda > 0$, and $t < 0$ (we can assume that the perturbation arrives at the centre of symmetry $t = 0$), the same integral curves describe the flows characterized by a constant velocity of the advancing front of a weak perturbation in the case of converging ($v < 0$) flows.

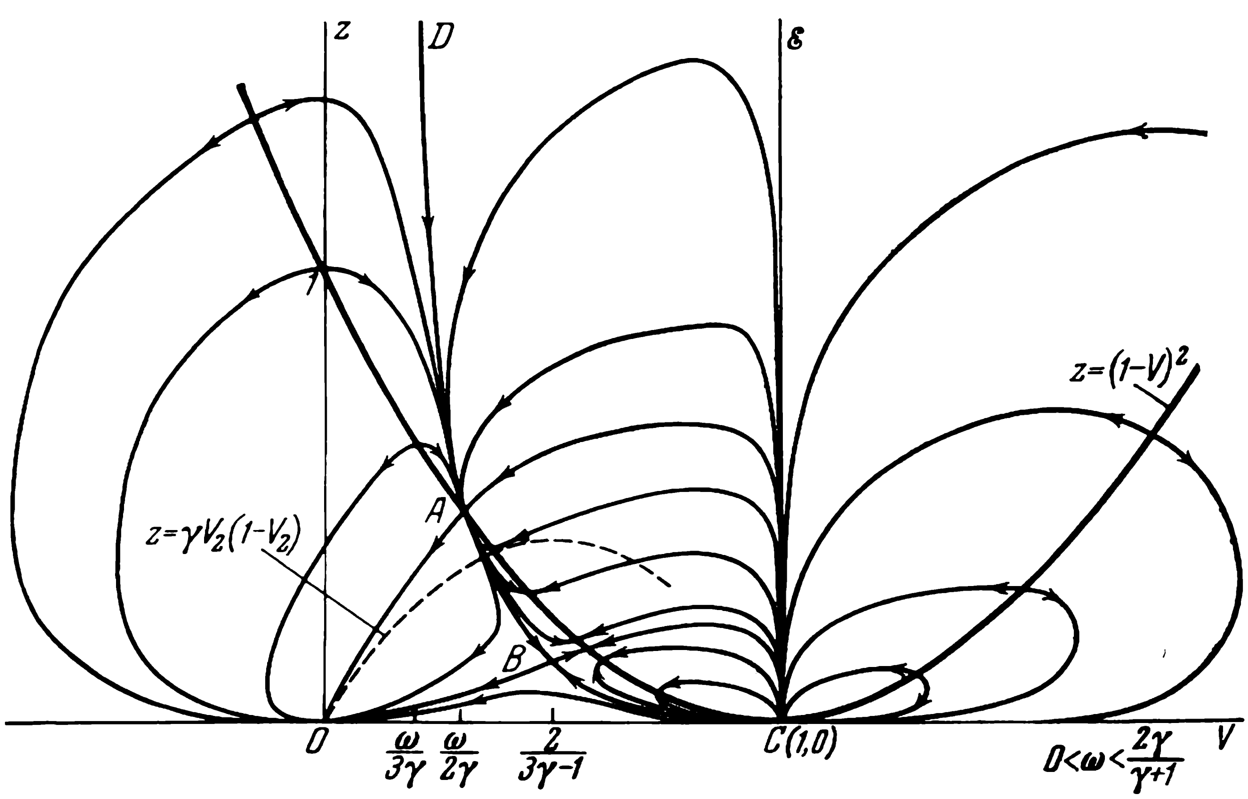

Fig. 39. The family of integral curves for $\nu = 3$, $\delta = 1$, and $0 < \omega < 2\gamma/(\gamma + 1)$ (the point A lies above the parabola $z = \gamma V_2 (1 - V_2)$). We note that the singular point A is shifted along the parabola $z = (1 - V)^2$, the point D is also displaced, and the property of the singular point C is modified.

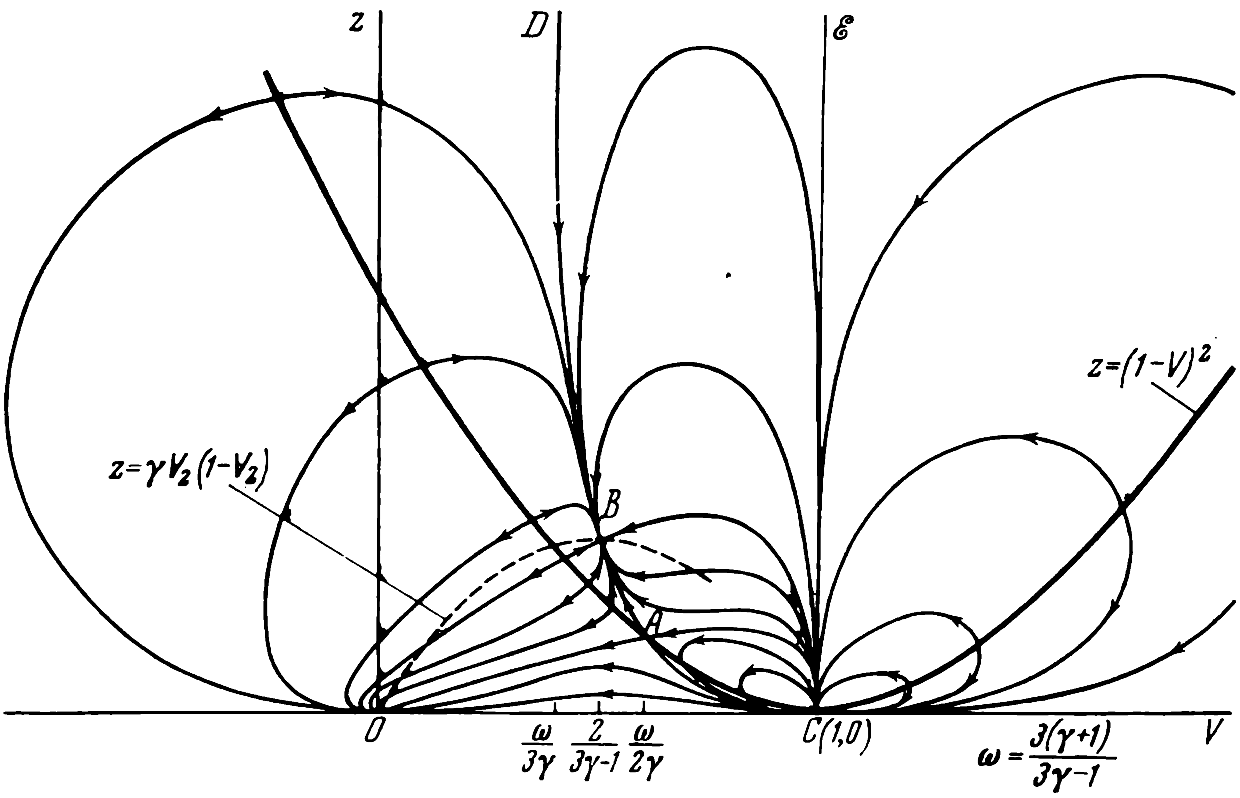

Fig. 40. The family of integral curves for $\delta = 1$ and $\omega = 3(\gamma + 1)/(3\gamma - 1)$. The singular point B lies on the parabola $z = \gamma V_2(1 - V_2)$.

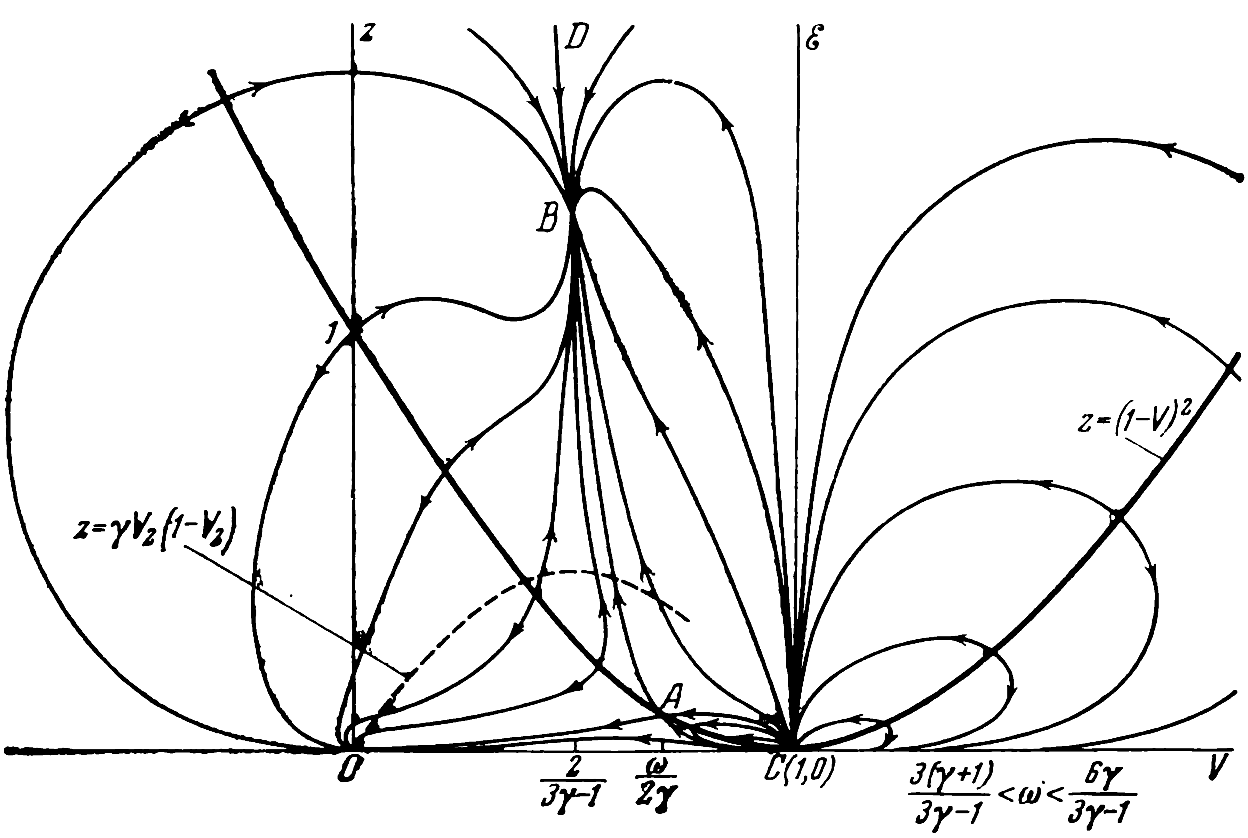

Fig. 41. The family of integral curves for $\nu = 3$, $\delta = 1$, and $3(\gamma + 1)/(3\gamma - 1) < \omega < 6\gamma/(3\gamma - 1)$. The motion toward the singular point C corresponds to the motion toward the centre of symmetry from points of the parabola $z = \gamma V_2(1 - V_2)$.

formulas hold for the integral curves near the point O:

$$z = CV^2, \quad \lambda = \frac{C_1}{V} \quad \text{and} \quad z = \frac{\gamma}{\omega} V, \quad \lambda = \frac{C_1}{\sqrt{V}} \tag{5.6}$$

The point O in the (z, V) plane corresponds to points at infinity in the gas.

The point C ($z = 0$, $V = 1$) is a multiple node; one set of integral curves touch the V-axis at C in the direction perpendicular to this axis. When $\omega > 0$, a second set of integral curves touch the line $V = 1$ at C.

The asymptotic formulas for the integral curves touching the V-axis for $\gamma < 2$ are

$$\left.\begin{aligned} &\lambda - \lambda^* = \frac{\gamma+1}{3(\gamma-1)}\lambda^*(1-V), \quad z = \frac{2(2-\gamma)}{6 - \frac{\gamma+1}{\gamma}\omega}(1-V)^2 \\ &\mathscr{R} = C(1-V)^{\frac{6-\omega-\gamma\omega}{3(\gamma-1)}}, \quad P = C\frac{2(2-\gamma)}{6\gamma-(\gamma+1)\omega}(1-V)^{\frac{6\gamma-\omega(\gamma+1)}{3(\gamma-1)}} \end{aligned}\right\} \tag{5.7}$$

where C and λ^* are certain constants; the quantity $P \to 0$ as $\lambda \to \lambda^*$ for $\omega < 6\gamma/(\gamma + 1)$.

The following asymptotic formulas are valid for the curves perpendicular to the V-axis:

$$\left.\begin{aligned} &\lambda - \lambda^* = \frac{\gamma}{\nu\gamma - \omega}(1-V) \quad \text{for} \quad \omega < 6\gamma/(\gamma+1) \\ &z = A(1-V)^{\frac{(\gamma-1)\omega}{3\gamma-\omega}}, \quad \mathscr{R} = B(1-V)^{\frac{-(\gamma-1)\omega}{3\gamma-\omega}}, \quad P = \frac{AB}{\gamma} \end{aligned}\right\} \tag{5.8}$$

where λ^*, A, and B are certain constants. As we move through the gas to the point C, we approach a surface at a finite distance from the centre, a piston, or the boundary of a vacuum; at this distance the phase velocity equals the gas particle velocity.

The point

$$B\left(z = \frac{18\gamma(\gamma-1)^2}{(3\gamma-1)^2[\omega+3\gamma(2-\omega)]}, \quad V = \frac{2}{3\gamma-1}\right)$$

is a node for $\omega > 4\gamma/(3\gamma - 1)$ and a saddle point for $\omega < 4\gamma/(3\gamma - 1)$; for $\omega = 6\gamma/(3\gamma - 1)$ the point B coincides with the singular point at infinity D. The point B transforms into the lower half-plane for large ω. If the point B does not lie on the parabola $z = (1 - V)^2$, then the variable λ on the integral curves entering at this point approaches zero or infinity, i.e. the centre of symmetry or the point at infinity. If the point B lies on the parabola $z = (1 - V)^2$, which is possible only for $\omega = 4\gamma/(3\gamma - 1)$, then the value of λ in this case remains finite as the point B is approached. In this case, the

point B can correspond to a moving point in the gas. If

$$\omega = \frac{3(\gamma+1)}{3\gamma-1}$$

then the point B lies on the parabola $z_2 = \gamma V_2(1 - V_2)$; for smaller ω it lies below and for larger ω it lies above this parabola. This result is of importance in solving problems of propagation of detonation in a medium with density varying as $\rho_1 = A/r^\omega$.

The point D ($z = \infty$, $V = \omega/3\gamma$) is a saddle point for $\omega < 6\gamma/(3\gamma - 1)$ and a node for $6\gamma/(3\gamma - 1) < \omega < 3\gamma$. In the first case, there is one integral curve that has the z-axis as an asymptote; $\lambda \to 0$ along this curve as $z \to \infty$ and, therefore, as we approach the centre of symmetry. In the second case, we withdraw toward infinity as the node D is approached along the integral curves, since $\lambda \to +\infty$. The following asymptotic formulas hold near the point D for $\omega < 6\gamma/(3\gamma - 1)$:

$$\left.\begin{aligned}
\lambda &= C\left(V - \frac{\omega}{3\gamma}\right)^{\frac{3\gamma-\omega}{6\gamma+\omega-3\gamma\omega}}\\
z &= \frac{\omega(3\gamma-\omega)^3}{27\gamma^3(15\gamma-3\gamma\omega-2\omega)}\,\frac{1}{V-\frac{\omega}{3\gamma}}\\
\mathscr{R} &= B\left(V-\frac{\omega}{3\gamma}\right)^{\frac{\omega(3-\omega)}{6\gamma+\omega-3\gamma\omega}} = B_1\lambda^{\frac{\omega(3-\omega)}{3\gamma-\omega}}\\
P &= B_1 C^{\frac{6\gamma+\omega-3\gamma\omega}{3\gamma-\omega}}\,\frac{\omega(3\gamma-\omega)^3}{27\gamma^4(15\gamma-3\gamma\omega-2\omega)}\,\lambda^{\frac{3\gamma\omega+2\omega-\omega^2-6\gamma}{3\gamma-\omega}}
\end{aligned}\right\} \qquad (5.9)$$

where B_1 and C are arbitrary constants; B is expressed in terms of B_1 and C.

The asymptotic formulas for velocity, pressure, and density are easily obtained near the centre of symmetry from (5.9) and from basic formulas (1.3).

The point A $\left[z = \left(1 - \frac{\omega}{2\gamma}\right)^2;\ V = \frac{\omega}{2\gamma}\right]$ is a focus for $\omega < 0$, a node for $0 \leqslant \omega < 4\gamma/(3\gamma - 1)$, a saddle point for $4\gamma/(3\gamma - 1) < \omega < 2\gamma$, a node for $2\gamma < \omega < 4\gamma(\gamma + 1)/(6\gamma - 2 - \gamma^2)$, and a centre for $4\gamma(\gamma + 1)/(6\gamma - 2 - \gamma^2) < \omega$.

The integral curves approach the node tangentially to the z-axis for $\omega = 0$.

There are two directions in which the integral curves may approach the point A for $0 < \omega < 4\gamma(\gamma + 1)/(6\gamma - 2 - \gamma^2)$; the corresponding slopes along these directions are given by

$$k_{1,2} = \gamma\left(\frac{\omega}{2\gamma} - 1\right)\left[1 \pm \sqrt{\frac{\omega(\gamma^2 - 6\gamma + 2) + 4\gamma(\gamma+1)}{\omega\gamma^2}}\right]$$

The point A is always on the parabola $z = (1 - V)^2$; consequently, the variable λ at the point A must be finite. Passing through the point A where the phase and particle velocities are different corresponds to crossing a characteristic; consequently, this point may correspond to a weak discontinuity.

The point $\mathscr{E}$ ($z = \infty$, $V = 1$) is a node for $\omega < 0$ and a saddle point for $0 < \omega < 3\gamma$.

The point G ($z = \infty$, $V = \infty$) is a saddle point for any ω.

A simple exact solution of the equations of gas dynamics for which $z =$ const and $V =$ const corresponds to each singular point; the variable λ remains free. The function $\mathscr{R}(\lambda)$ is determined from (5.5); it is found that $\mathscr{R}$ is a power function of λ for $C_2 = 0$ and, therefore, P is also a power function. Hence, this is a particular solution for which v, ρ, and p are power monomials of r and t. In particular, if $z = 0$ and $V = 0$, then we have a state of rest with zero pressure, which is the initial state in certain self-similar motions.

The qualitative pictures of the family of integral curves are given in Figs. 38-41 for $\gamma = 5/3$ in the cases

$$\omega = 0, \quad 0 < \omega < \frac{2\gamma}{\gamma+1}, \quad \omega = \frac{3(\gamma+1)}{3\gamma-1}, \quad \text{and} \quad \frac{3(\gamma+1)}{3\gamma-1} < \omega < \frac{6\gamma}{3\gamma-1}$$

The direction of increasing λ is shown by arrows. The parameter λ attains a maximum or a minimum on the parabola $z = (1 - V)^2$; consequently, a continuous passage along the integral curve across this parabola is impossible since it will generate two sheets in the space of gas motion, and the solution will not be unique. However, the passage along the integral curve across the parabola $z = (1 - V)^2$ is possible if this curve passes through the singular point A on the parabola. A change from a maximum to a minimum for λ occurs at A on the parabola. When moving along the intersecting integral curve, the parameter λ has a finite value at A and varies monotonically. We shall see below that this fact isolates the intersecting integral curve as a solution of the appropriate detonation problem.

We can have $p_1 \neq 0$ for $\omega = 0$ in self-similar solutions: the whole line $V = 0$ then corresponds to a state of rest.

The parameter λ must vary from 0 to ∞ along the integral curves in the solutions being investigated if the gas is unbounded. An appropriate motion is possible in the majority of cases only with strong shocks.

The initial point $\lambda = 0$ and the final point $\lambda = \infty$ can only correspond to the above singular points; the piston or vacuum boundaries can only correspond to points on the line $V = 1$ (where the phase and particle velocities are identical), and, in particular, to the singular points C and $\mathscr{E}$.

One of the two variables, pressure or density, is either zero or infinity at the points C and $\mathscr{E}$. Cases of zero pressure correspond to

a vacuum, cases of finite pressure and infinite density at the point C and zero density at the point $\mathscr{E}$ correspond to the motion of a spherical piston.

Let us consider the family of integral curves of differential equation (2.1) for case 2. We have

$$\frac{dz}{dV} = \frac{z\,[2(V-1)+\nu(\gamma-1)V](V-\delta)}{\left[V(V-1)(V-\delta)+\nu\left(\frac{\delta}{\gamma}-V\right)z\right]} - \frac{z\left\{(\gamma-1)V(V-1)(V-\delta)+\left[2(V-1)+\frac{\nu\delta}{\gamma}(\gamma-1)\right]z\right\}}{(V-\delta)\left[V(V-1)(V-\delta)+\nu\left(\frac{\delta}{\gamma}-V\right)z\right]} \tag{5.10}$$

$$\frac{d\ln\lambda}{dV} = \frac{z-(V-\delta)^2}{V(V-1)(V-\delta)+\nu\left(\frac{\delta}{\gamma}-V\right)z} \tag{5.11}$$

where $\delta = 2/(2+\nu-\omega)$. The third equation defining $\mathscr{R}$ can be replaced by the entropy invariant. We obtain from (3.9) for $\delta = 2/(2+\nu-\omega)$, $s = 0$, $k = \omega - 3$, and from (3.7) for $\omega < \nu$:

$$z = \mathscr{R}^{\frac{2\nu-\nu\gamma-\omega}{\omega-\nu}}\left(V-\frac{2}{2+\nu-\omega}\right)^{\frac{\nu-\gamma\omega}{\omega-\nu}}\lambda^{-(2+\nu-\omega)}C \tag{5.12}$$

where C is a constant of integration and the condition $\omega < \nu$ can be replaced by the inequalities $\omega \neq \nu$ and $\omega \neq \nu + 2$. The invariant defining the mass

$$\mathscr{M} = \sigma_\nu \int_0^r \frac{A}{r^\omega} r^{\nu-1}\,dr$$

diverges for $\omega \geqslant \nu$. The constant on the right-hand side of (3.7) is taken to be zero on the basis of the remarks made on page 190.

Relation (5.12) defines $\mathscr{R}$ in terms of z, V, and λ.

The constant E with the dimensions $ML^{\nu-1}T^{-2}$ is a characteristic constant; consequently, an energy invariant holds in case 2. We obtain on the basis of (3.11)

$$\lambda^{\nu+2}\left[zV+\left(V-\frac{2}{2+\nu-\omega}\right)\left(\frac{\gamma V^2}{2}+\frac{z}{\gamma-1}\right)\right]\mathscr{R} = C_1 \tag{5.13}$$

Invariant (5.13) can replace equation (5.10). It is possible to determine z from (5.13) and to substitute it into (5.11); we then obtain an ordinary equation containing only λ and V. If $C_1 = 0$, then (5.13) yields an integral of (5.10), and (5.11) is integrated by quadrature. If $C_1 \neq 0$, then (5.10) can be integrated to determine $z(V)$ and afterwards the function λ can be calculated by using (5.13) without integrating (5.11).

A qualitative picture of the family of integral curves in the (z, V)-plane is shown in Fig. 42 for (5.10) with $\nu = 3$, $\omega = 0$, and $\gamma < 2$. The direction of increasing λ is shown by arrows. The character of the singular points is seen in the schematic.

We note that the singular point C $(z = \infty,\ V = 2/(2 + \nu - \omega)\,\gamma)$ is a saddle point. Along a single integral curve terminating at a point, the variable λ tends to zero as the point C is approached. It

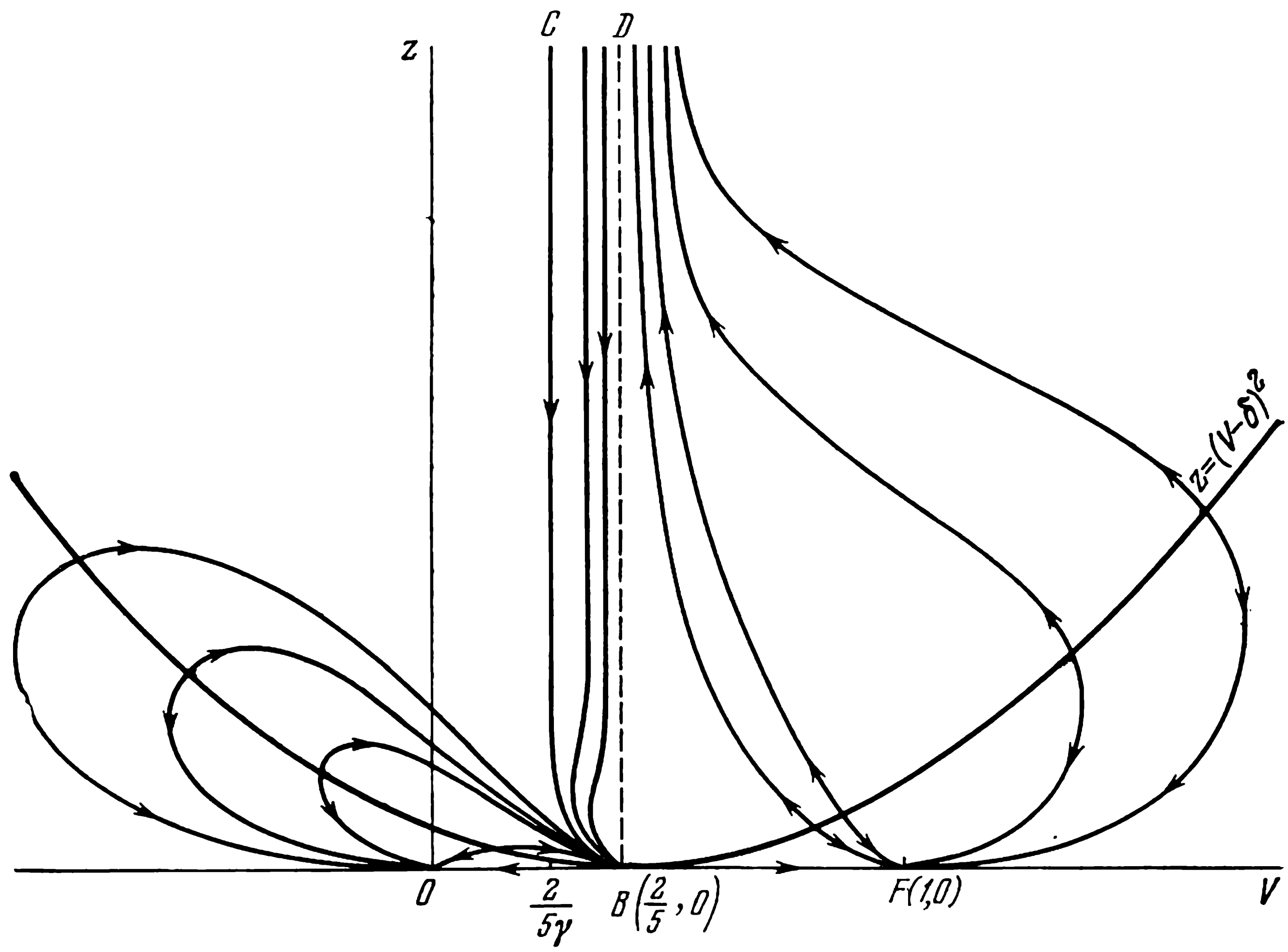

Fig. 42. The family of integral curves for $\nu = 3$, $\omega = 0$, $\delta = 2/5$, and $\gamma < 2$.

should be noted that we obtain the equation of this curve from (5.13) for $C_1 = 0$.

When analysing the family of integral curves in the (z, V) plane as a function of ω and γ, it is necessary to take into account the fact that (5.10) has the singular point

$$\mathscr{E}\left(V^* = \frac{2}{[2+\nu(\gamma-1)]},\quad z^* = -\frac{2(\gamma-1)\gamma[(2-\gamma)\nu-\omega]}{[2+\nu(\gamma-1)]^2[(2-\omega)\gamma+\nu-2]}\right) \tag{5.14}$$

which can pass through the point B from the lower half-plane $z < 0$ to the upper half-plane $z > 0$. It is easy to verify that the singular point $\mathscr{E}$ lies on the integral curve passing through the singular point C, i.e. z^* and V^* satisfy (5.13) for $C_1 = 0$.

The condition that the coordinate z^* be positive is

$$\nu(2-\gamma) \leqslant \omega \leqslant \frac{2\gamma+\nu-2}{\gamma} \tag{5.15}$$

The point $\mathscr{E}$ coincides with the point B at the lower bound in (5.15) and with the point C at the upper bound.

In Fig. 43 the family of integral curves is shown for the case when $\mathscr{E}$ is in the upper half-plane.

Only the point C and the singular points $z = 0$ and $V = \pm\infty$ can correspond to the centre of symmetry $\lambda = 0$.

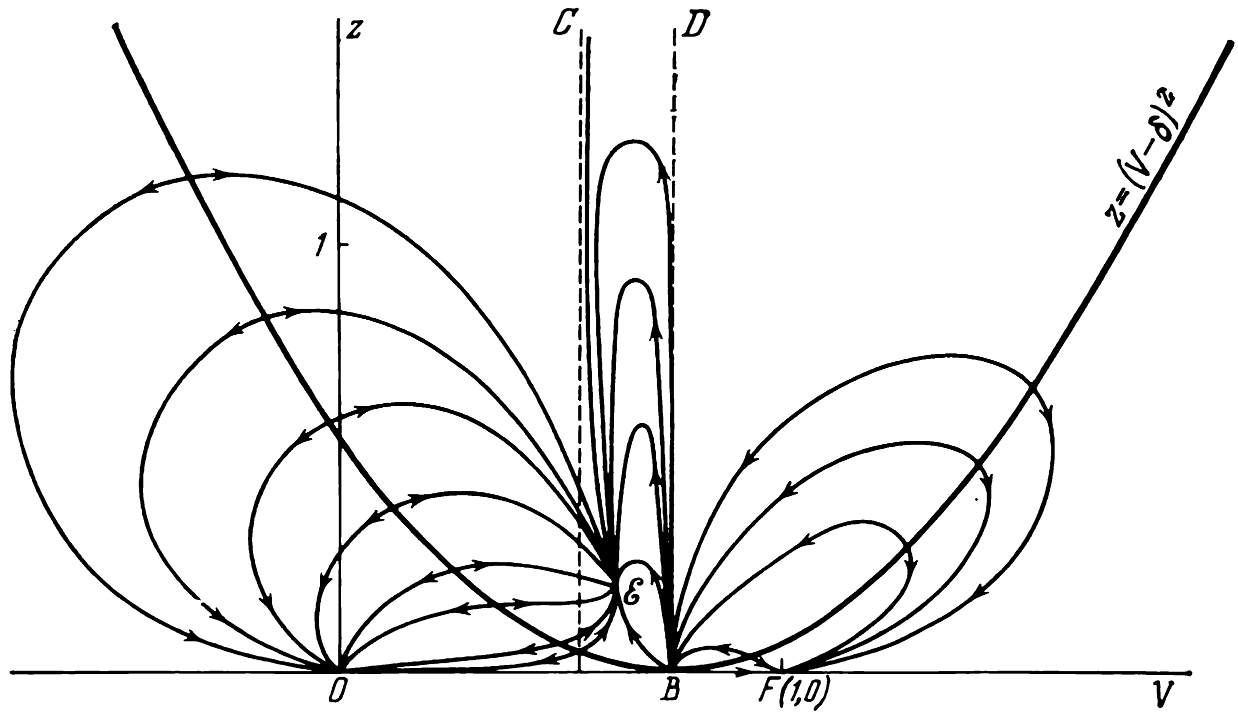

Fig. 43. The family of integral curves for $\nu = 3$, $6 - 3\gamma < \omega < (2\gamma + 1)/\gamma$, and $\delta = 2/(5 - \omega)$. The singular nodal point $\mathscr{E}$ appears in the upper half-plane. The field corresponds to the case $\omega > 3/\gamma$.

The singular points O, F, and $\mathscr{E}$ correspond to the infinitely remote points $\lambda = \infty$.

The parameter λ has a finite value at the singular points B and D which correspond to the boundaries of an expanding spherical piston or the boundary of a vacuum.

§ 6. The Piston Problem

We now analyse the perturbation at a certain instant t in a gas forced out by a spherical piston expanding at a constant velocity U. The initial pressure p_1 and the initial density ρ_1 are constant and nonzero [21].

Since $v = U$ and $r = Ut$ for the fluid particles adjacent to the piston, then

$$V = \frac{v}{r/t} = 1$$

on the piston.

Therefore, the image point in the (z, V) plane corresponding to the piston (Fig. 44) must be on the line $V = 1$. The motion along

the integral curve in the directions of increasing λ corresponds to the motion along the radius from the piston to infinity. The integral curve intersects the parabola $z = (V - 1)^2$. Since a continuous passage through this curve is impossible, the extension of the motion to the point O, which corresponds to the point at infinity, can only be achieved by means of a jump.

On the other hand, the gas is at rest in the unperturbed region, i.e. a point on the z-axis corresponds to the far (outer) side of the jump.

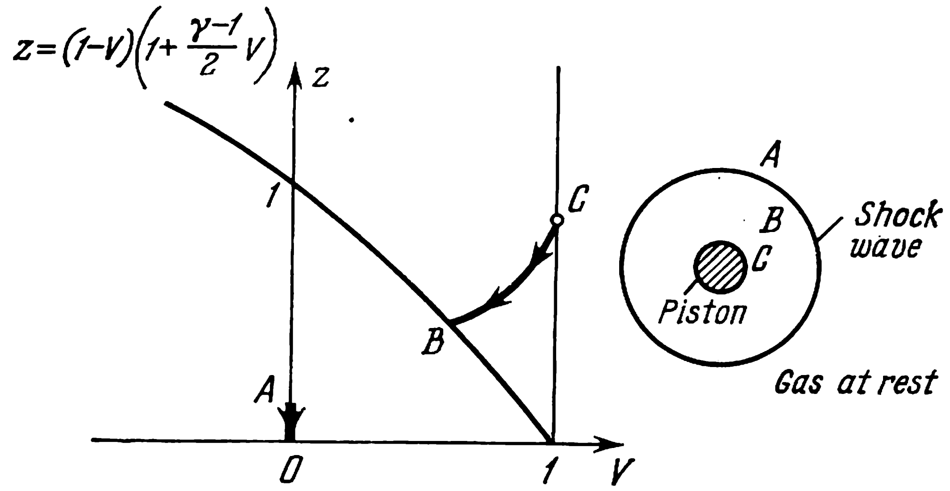

Fig. 44. The integral curve in the (z, V) plane for the solution of the spherical piston problem. The passage from the point A to the point B occurs by a jump across the shock wave. The point C corresponds to the piston. The curve BC corresponds to the adiabatic compression between the piston and the shock wave.

As was shown above, points of the z-axis transform discontinuously into points of the parabola

$$z = (1 - V)\left(1 + \frac{\gamma - 1}{2} V\right) \tag{6.1}$$

Thus the image point for the motion between the piston and infinity in the physical space moves along an integral curve from a certain point on the line $V = 1$ to the intersection with parabola (6.1), and then jumps to the line $V = 0$ discontinuously (see Fig. 44).

The physical interpretation of this motion is that a shock wave is propagated through a gas at rest followed by the adiabatic compression of the gas between the shock wave and the piston.

The case of a cylindrical piston does not differ qualitatively from that of a spherical piston. It is evident that, in contrast to the spherical case, a region of a gas moving at a constant velocity, density, and pressure exists between the shock wave and the piston for a plane piston moving in a cylindrical pipe.

The curves in Figs. 45 and 46 give the ratio of the gas pressure and density, respectively, at the piston to the corresponding values of the gas at rest as a function of the ratio of the piston velocity to the initial speed of sound.

Figure 47 shows the ratio of the shock wave velocity to the speed of sound as a function of the ratio of the piston velocity to the speed of sound.

It appears from the graphs that for equal piston velocities the gas compression is larger in the plane case than in the spherical case.

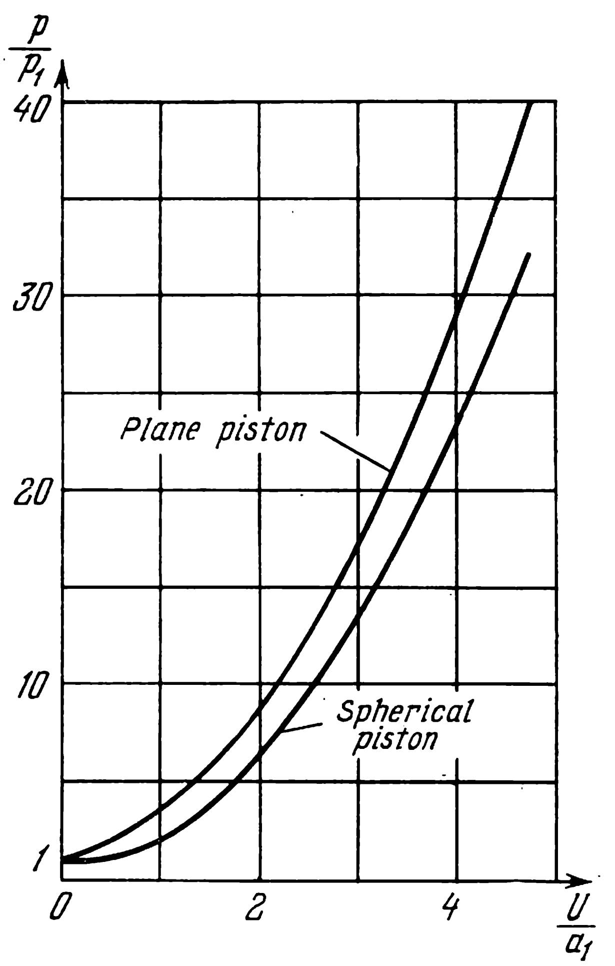

Fig. 45. The pressure p on a piston as a function of the piston velocity U; p_1 is the pressure and a_1 is the speed of sound in the unperturbed gas.

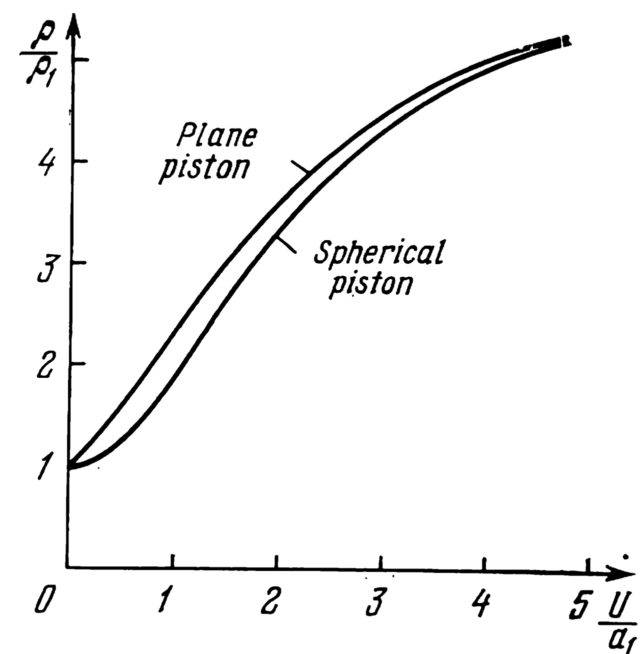

Fig. 46. The density ρ of particles adjacent to the piston as a function of the piston velocity U; ρ_1 is the density and a_1 is the speed of sound in the unperturbed gas.

The shock wave is stronger in the plane case than in the spherical case (especially for low piston velocities).

In the plane case, in contrast to the spherical and cylindrical cases, one can consider the problem of a piston being drawn away from the gas (backstroke) in addition to the problem of gas compression by the piston. The solution of this problem is easily constructed by using the translational motions and the singular solution of system of equations (1.3) for $\nu = 1$ which, in the z, V, and λ variables, takes the form

$$\left.\begin{aligned} z &= (1-V)^2 \\ V - \frac{2}{\gamma-1}(1-V) &= C\lambda^{-1} \end{aligned}\right\} \qquad (6.2)$$

where C is a constant of integration.

After changing to dimensional variables, we obtain a nonlinear solution for a progressive suction wave in a gas (the most simple case of the Riemann solution):

$$\left.\begin{aligned} x &= (v \pm a)\,t \\ v &= \pm\frac{2}{\gamma+1}a + \text{const} \end{aligned}\right\} \qquad (6.3)$$

where a is the speed of sound [22].

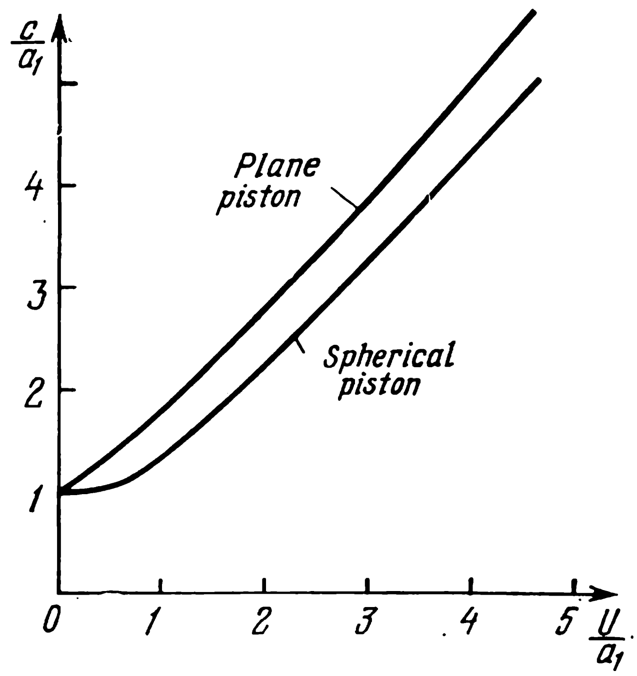

Fig. 47. The shock wave velocity c ahead of the piston as a function of the piston velocity U; a_1 is the speed of sound in the unperturbed gas.

§ 7. Problem of Implosion and Explosion at a Point

We consider the implosion and explosion problem in the case when the initial velocity, density, and pressure are everywhere uniform, i.e. $\omega = 0$ and $\delta = 1$. The appropriate family of integral curves in the (z, V) plane is given in Fig. 38.

The point O corresponds to the point at infinity. From (5.6), the asymptotic formulas near the point O are:

$$z = CV^2, \quad \lambda = \frac{C_1}{V}$$

We obtain from the condition at infinity or the initial condition:

$$z = \frac{\gamma p_1}{\rho_1} : \frac{r^2}{t^2} = C\frac{t^2}{r^2}v_1^2, \quad \text{whence} \quad C = \frac{\gamma p_1}{\rho_1 v_1^2} \qquad (7.1)$$

When $v_1 < 0$, we follow the left-hand branch of the parabola $z = CV^2$ with $V < 0$, when $v_1 > 0$, we follow the right-hand branch with $V > 0$.

Let us put $\lambda = r/(a_1 t)$ and $a_1 = \gamma p_1/\rho_1$; according to (5.2), we thus fix the constant β.

Using this, the asymptotic formula for λ yields:

$$\frac{r}{a_1 t} = \frac{C_1 r}{v_1 t}, \quad C_1 = \frac{v_1}{a_1} \qquad (7.2)$$

If the initial velocity is directed toward the centre (implosion), i.e. $v_1 < 0$, then the motion along the integral curve starting at O for negative V corresponds to the motion from infinity toward the centre for fixed t.

Since the gas velocity at the centre is zero for $t \neq 0$, then the centre can be reached either by moving along the integral curve

terminating at the singular point B or by moving along the integral line $V = 0$ terminating at the singular point D ($z = \infty$, $V = 0$). But it is impossible to pass from the region $V < 0$ where the image point is found to the integral curve terminating at the point B, and it is only possible to cross to the z-axis discontinuously.

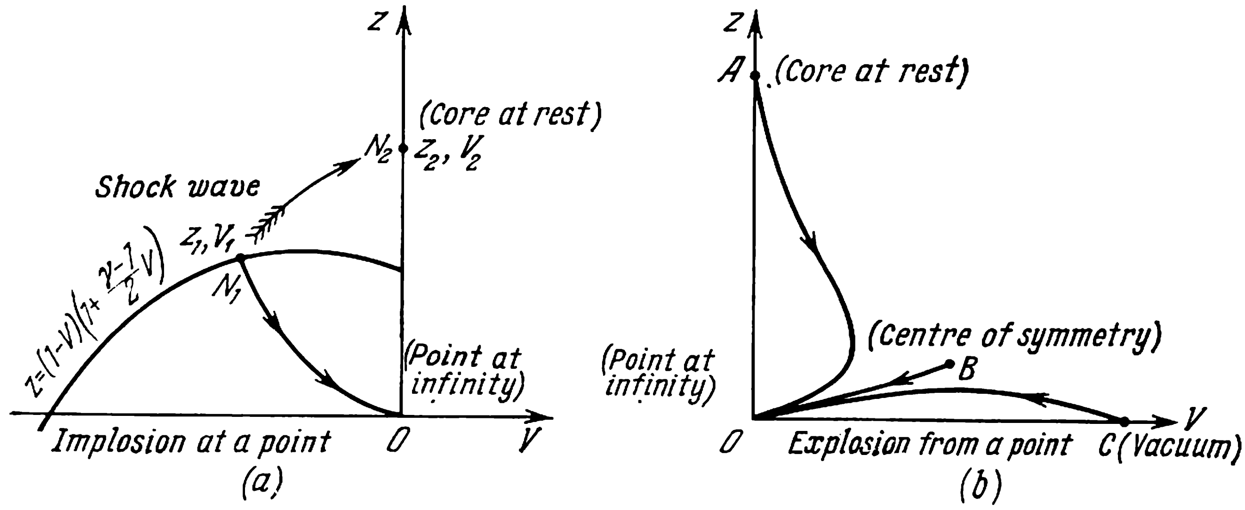

Fig. 48. The integral curves in the (z, V) plane corresponding to: (a) implosino at a point; (b) explosion from a point.

Therefore, the image point follows the integral curve until the intersection with parabola (2.11) is reached:

$$z = (1 - V)\left(1 + \frac{\gamma - 1}{2} V\right)$$

with the points of the z-axis transforming discontinuously to this parabola. After intersecting the parabola, the image point transforms into a certain point of the z-axis by a jump. The shape of the integral curve is shown in Fig. 48a.

Hence, in the physical space, the gas is first compressed adiabatically in its motion from infinity toward the centre, and is afterward brought to rest by a shock (Fig. 49a).

For low values of the initial velocity, the image point in the explosion case ($v_1 > 0$ is the gas particle velocity directed away from the centre) moves along the integral curve defined by (7.1) from the point O to the point A, and then along the integral line $V = 0$ (Fig. 48b). In the physical space, this corresponds to the motion from infinity toward the centre at the fixed instant of time accompanied by a drop in the gas density and pressure up to certain definite values; afterward, the gas is brought to rest by a weak shock (Fig. 49b). The singular point A (Fig. 48b) corresponds to the boundary of the spherical core of the gas at rest.

The integral curve OB corresponds to the solution in the (z, V) plane for a certain initial velocity $v = v_1^*$. The point B corresponds to the centre of symmetry. In this case, the velocity equals zero

only at the centre of symmetry in the physical space. If $v_1 > v_1^*$, then the integral curve can only be followed up to the singular point C where the parameter λ has a finite nonzero constant value λ^*, and the density and pressure equal zero. Therefore, a vacuum, propagating at a constant velocity defined by the value λ^*, is formed in the gas. By using the easily analysed family of integral curves

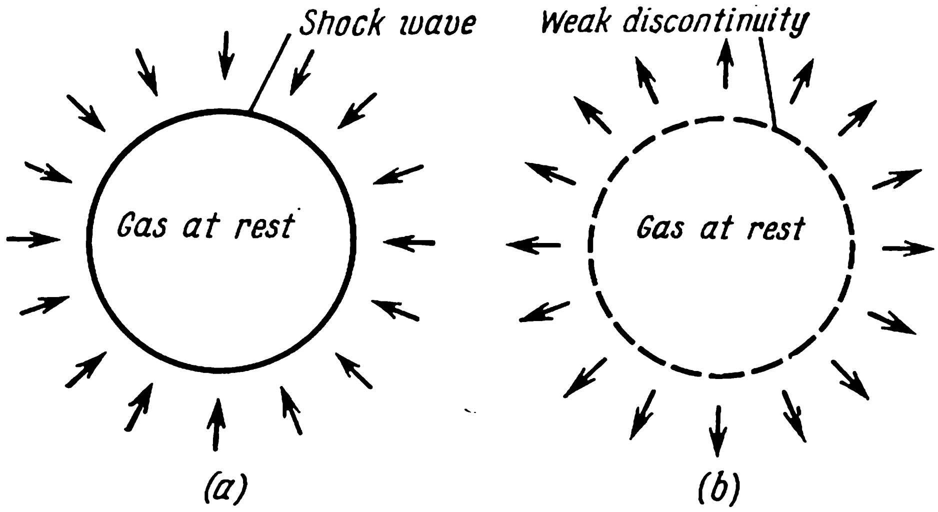

Fig. 49. Motion diagrams: (a) implosion; (b) explosion. The adiabatic compression or rarefaction arises in front of the core.

in the (z, V) plane for different δ, $\varkappa$ (or ω), ~~and γ~~, and the corresponding asymptotic formulas for the behaviour of the solutions in the vicinity of singular points, it is possible to establish the existence, qualitatively find the properties, and perform calculations for *all* different self-similar motions of the exploding or imploding gas in the time intervals $0 < t < +\infty$ and $-\infty < t < 0$. The solutions prove continuous or may contain discontinuities (strong or weak).

§ 8. Spherical Detonation

Let us consider the perturbation in a gas due to a detonation at the centre of symmetry at the time $t = 0$, a spherical detonation wave is propagated through the initially unperturbed gas when $t > 0$. We assume that heat is generated only at the shock front; the gas motion is adiabatic beyond the shock, which is a detonation wave [23].

Let us first study the case when the density ρ_1 and the pressure p_1 are constant and nonzero in the initially unperturbed gas. The perturbed motion of a perfect gas is determined by the parameters

$$r,\ t,\ \gamma,\ \gamma_1,\ p_1,\ \rho_1,\ Q$$

where Q is the heat liberated at the front by a unit mass of the gas;; γ_1 and γ are the appropriate values of the specific heat ratio (Poisson

constant): γ_1 ahead of the front and γ behind the front. The motion is self-similar and belongs to type 1 defined in § 5. The family of integral curves in the (z, V) plane for differential equation (5.3) $(\omega = 0)$ is shown in Fig. 38; $\lambda = \beta r/\sqrt{Qt}$.

The gas is at rest in the region ahead of the detonation wave; consequently, the point H_1 (Fig. 50) on the z-axis corresponds to the outer side of the detonation wave in the (z, V) plane. At this point $\lambda_2 = \beta c/\sqrt{Q}$; we determine β from the condition $\lambda_2 = 1$. It follows from the Hugoniot condition (2.27) that the inner side of the detonation wave in the (z, V) plane corresponds to the parabola

$$z_2 = (1 - V_2)^2 \frac{1+\gamma\Lambda}{1-\Lambda} \qquad (8.1)$$

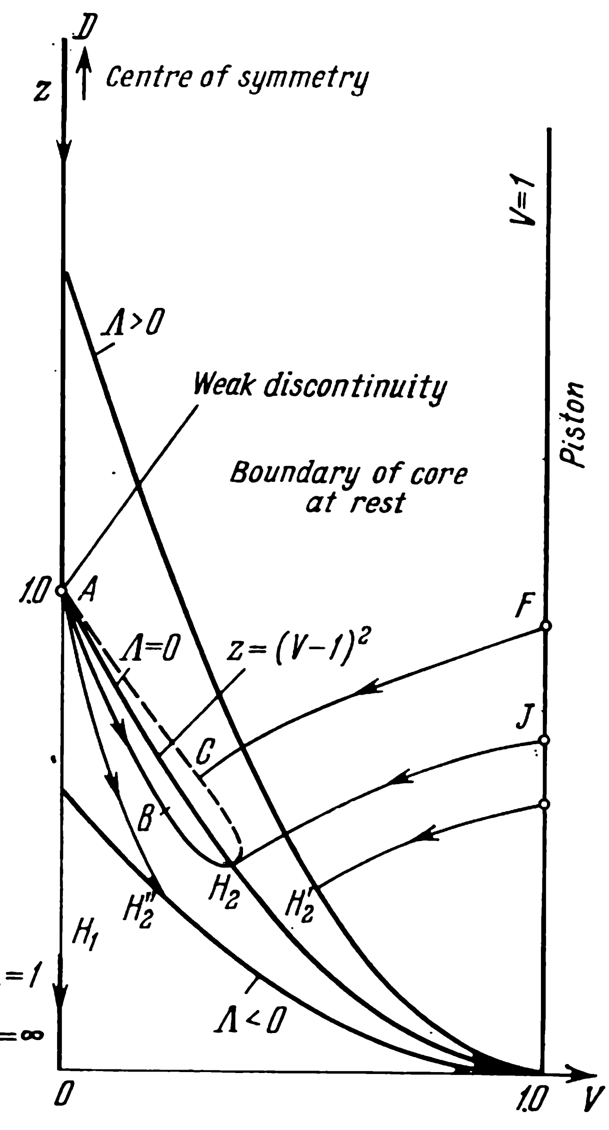

Fig. 50. The integral curves in the (z, V) plane corresponding to a spherical detonation. Case $p_1 = \text{const} \neq 0$ and $\rho_1 = \text{const}$ $(\omega = 0)$.

The Chapman-Jouguet condition is satisfied for $\Lambda = 0$, and parabola (8.1) coincides with the parabola

$$z = (1 - V)^2 \qquad (8.2)$$

If $\Lambda > 0$, parabola (8.1) lies above parabola (8.2) (see Fig. 38 and the diagram in Fig. 50). Since the variable λ has an extremum on parabola (8.2), it is impossible to extend the solution for $\Lambda > 0$ continuously between the points of parabola (8.1) and the centre of symmetry. It is also easy to see that a solution with an additional compression shock is impossible.

However, the solution can be continued up to the line $V = 1$; the points of this line can be considered to represent a spherical piston. Hence, the solution of the problem of the piston expansion in a detonated mass of a gas can be obtained.

The choice of the appropriate integral curve and of the parameter Λ is fixed by the values of the pressure or velocity (λ^*) on the piston since the point H_2' on parabola (8.1) (see Fig. 50) is determined for each Λ by the value of the parameter $p_1/(Q\rho_1)$.

If $\Lambda < 0$, then parabola (8.1) is situated below parabola (8.2). In this case, the solution of the problem is not unique.

There is a whole pencil of curves issuing from the node A which give the solutions satisfying all the boundary conditions. The position of a point behind the shock front H_2'' and the parameter $\lambda < 0$ are determined by finding the detonation front velocity. A core of a gas at rest is obtained at the centre whose boundary corresponds to the point A and constitutes a weak discontinuity. The line AD corresponds to the points of the core. The point D corresponds to the core centre $\lambda = 0$ (Fig. 51). If the point H_2 lies on parabola (8.2), then the Chapman-Jouguet condition is satisfied and the detonation velocity will be at a minimum.

If $\Lambda \leqslant 0$, then a suction wave extends from the rear of the detonation front to the core of the gas at rest.

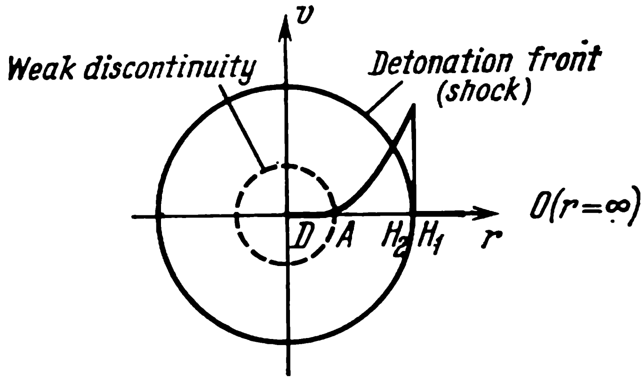

Fig. 51. Flow diagram for a spherical detonation. The qualitative pattern is the same as in the plane wave case (Grib, 1939).

The $\Lambda < 0$ cases correspond to the second root ρ_1/ρ_2 (see (2.16)). These can be eliminated owing to the additional property of monotonicity of the function Q' (ρ_1/ρ_2') in the zone of chemical reaction. If this function is nonmonotone in the zone of chemical reaction solutions of this type are admissible, and the selection of the required solution reduces to the determination of $\Lambda < 0$ by using the data on the kinetics of the chemical reactions and the available information on the nonmonotonicity of the function Q' (ρ_1/ρ_2'). On the other hand, we have mentioned above that in the $\Lambda > 0$ case the function Q' (ρ_1/ρ_2') is not necessarily monotone.

It must be mentioned that in the case of the Chapman-Jouguet mode ($\Lambda = 0$) two continuous solutions exist. In addition to the suction wave solution, there is the compression wave solution extended to the surface of the piston moving at an appropriate constant velocity U_J.

If the piston moves at a velocity less than U_J, the following solution can be constructed. The detonation wave propagating at the Jouguet velocity is coupled to a suction wave corresponding to the segment H_2B (see Fig. 50), a shock takes place from the point B to the point C, after which the solution is extended to the piston surface at F. The constant velocity U_F at the piston is less than U_J. The location of the point B is determined by the value of the velocity U_F.

We thus find that a unique self-similar solution can be constructed for any arbitrary piston velocity U_F [24].

Figures 52, 53, and 54 show the computed distributions of pressure, velocity, and temperature in an example in which $\gamma = \gamma_1 = 5/3$ and $p_1 = 0$ for satisfied Chapman-Jouguet condition (the curves corresponding to $\omega = 0$).

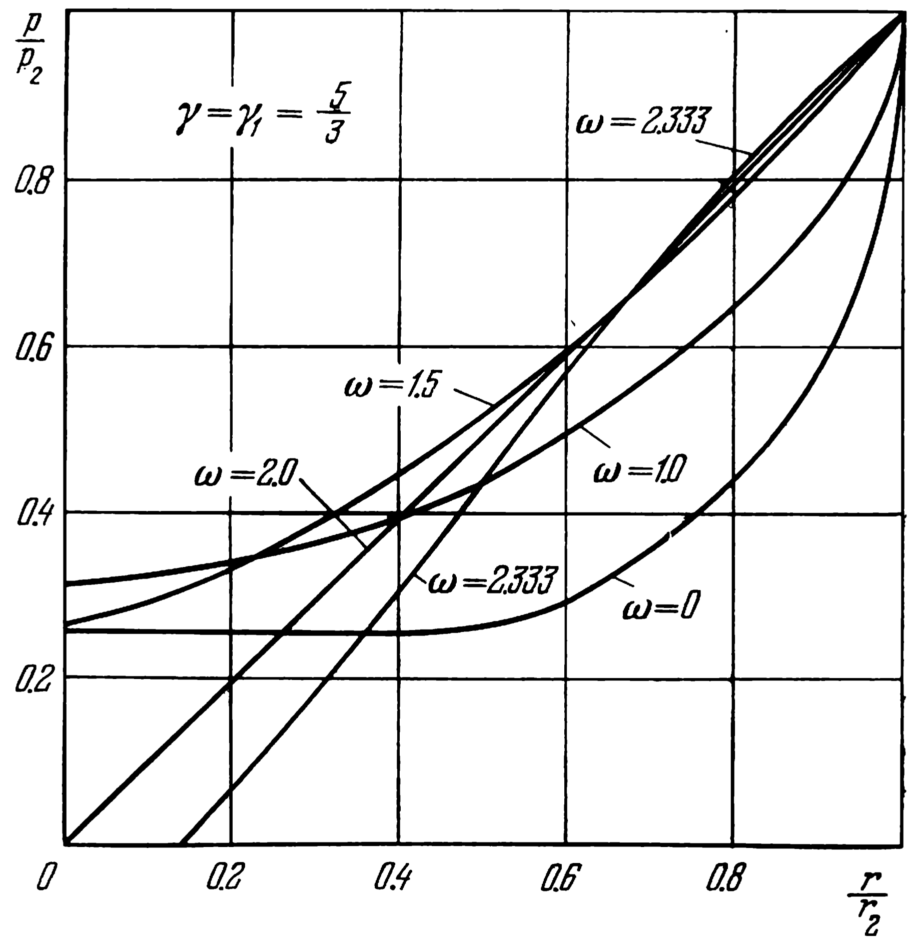

Fig. 52. Pressure distribution behind the detonation wave front; the initial pressure $p_1 = 0$, the initial density $\rho_1 = A/r^{\omega}$; a vacuum is formed near the centre, for $\omega > 3(\gamma + 1)/(3\gamma - 1)$, within which the pressure is zero.

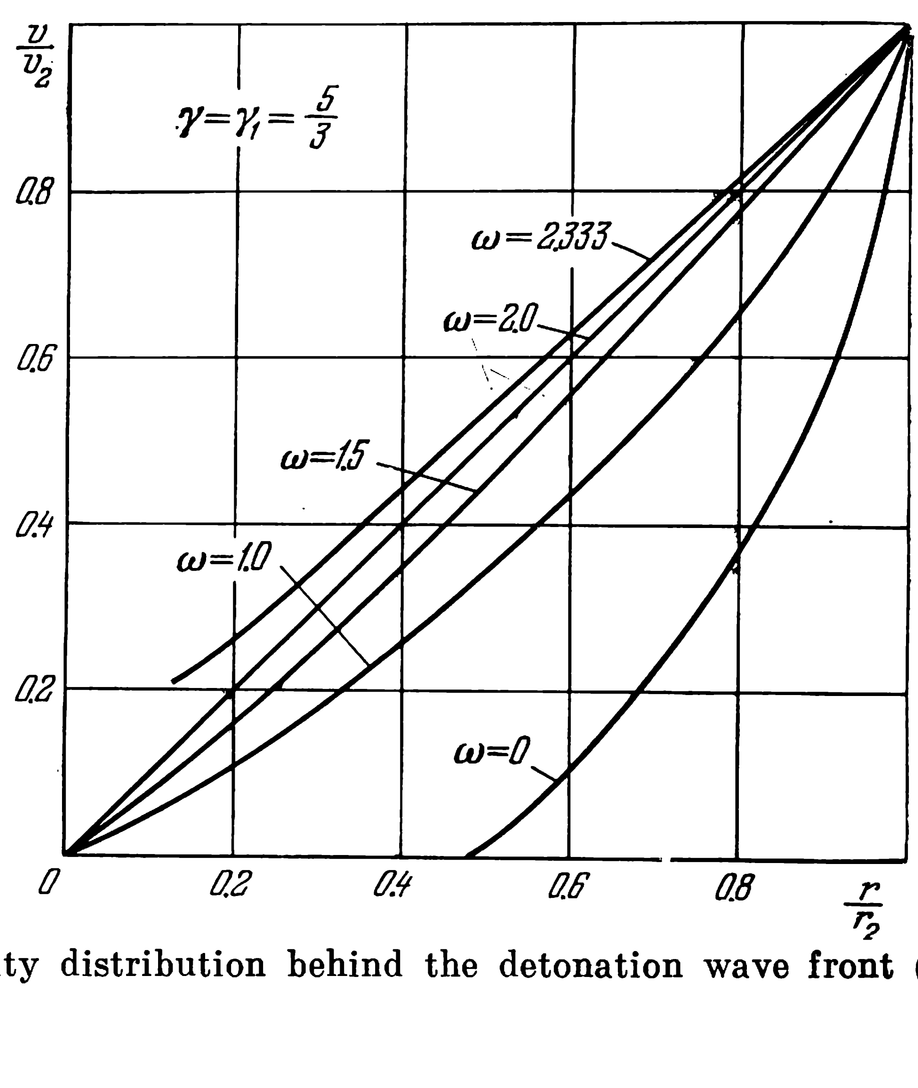

Fig. 53. Velocity distribution behind the detonation wave front ($p_1 = 0$ and $\rho_1 = A/r^{\omega}$).

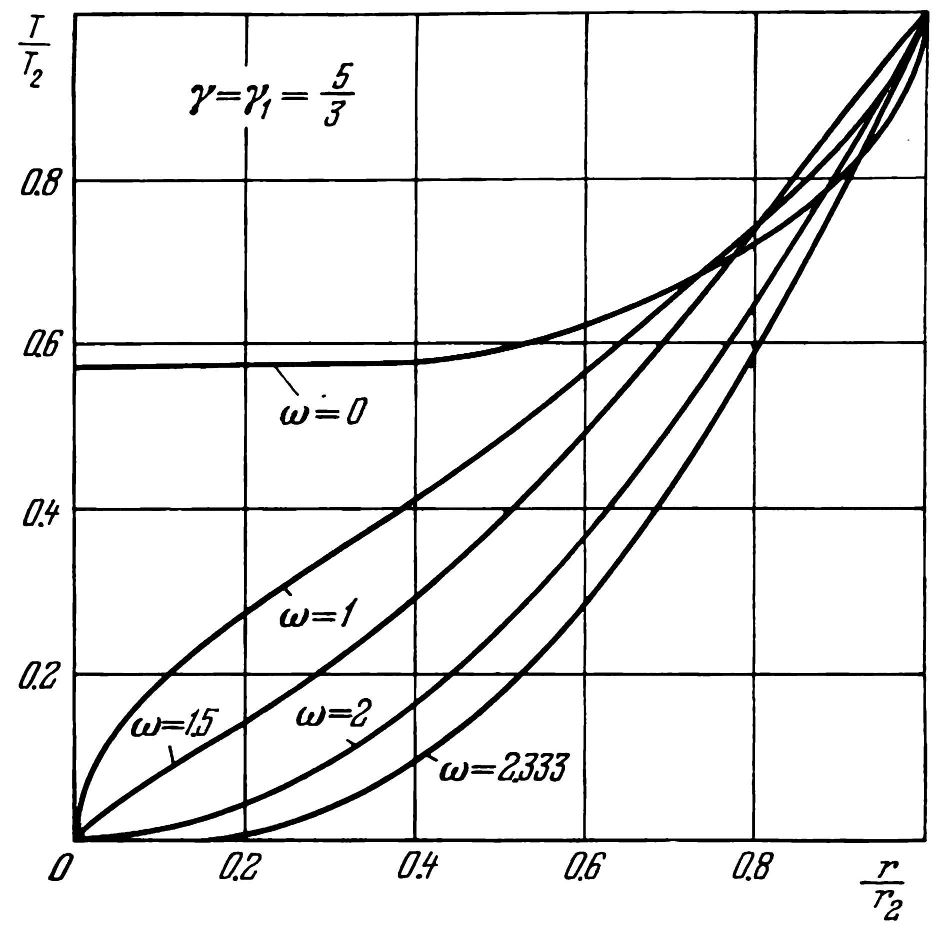

Fig. 54. Temperature distribution behind the detonation wave front ($p_1 = 0$ and $\rho_1 = A/r^{\omega}$).

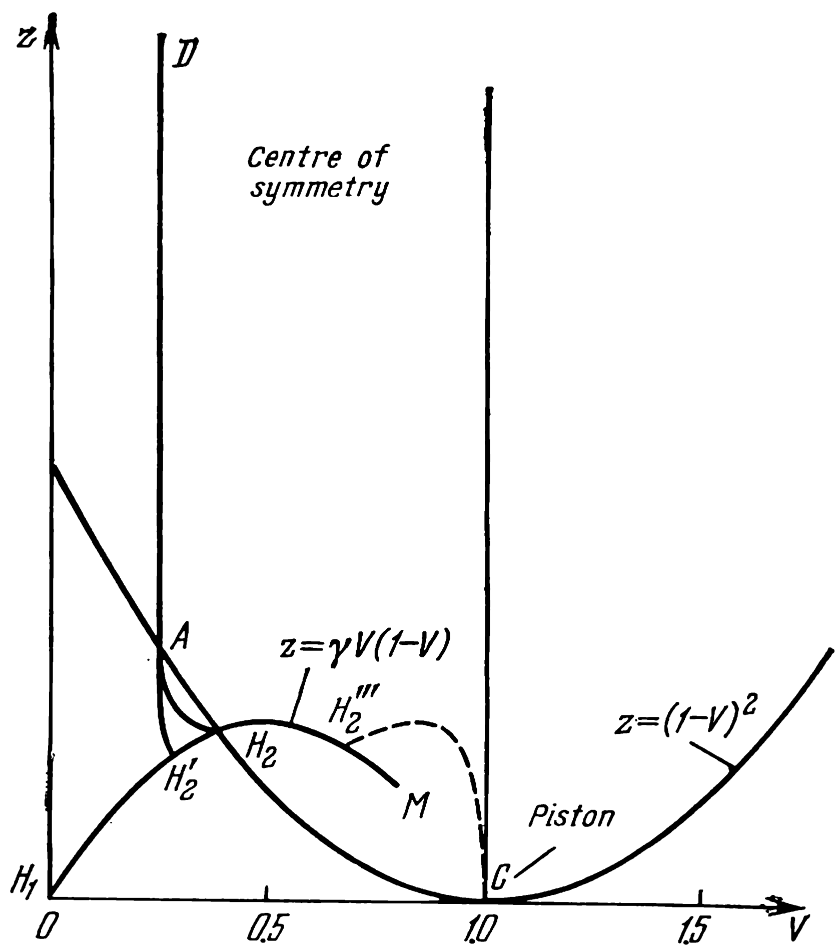

Fig. 55. The integral curves H_2AD or $H_2'AD$ correspond to the solution of the detonation problem for $0 < \omega < 2\gamma/(\gamma + 1)$. The Chapman-Jouguet condition is satisfied at the point H_2.

The solution of the detonation problem in the cylindrical and plane wave cases can be obtained in a similar manner [25].

We now consider the problem of detonation in a medium with a variable initial density [26]. If

$$0 < \omega < \frac{2\gamma}{\gamma+1} \quad \left(\rho_1 = k_1 \frac{A}{r^\omega}\right)$$

then the family of integral curves is shown in Fig. 39.

Figure 55 shows the integral curves in the (z, V) plane relevant to the problem of detonation wave propagation.

In order to continue the solution inward behind the detonation wave front, it is necessary to use the integral curve which starts from the point D at which $\lambda = 0$ and passes through the singular point A. A weak discontinuity can arise at the point A on the parabola

$$z = (1 - V)^2$$

the variable λ has a finite value and increases as we move downward along any integral curve; the only integral curves that yield a solution are those intersecting the parabola given by equation (2.29), namely,

$$z_2 = \gamma V_2 (1 - V_2) \tag{8.3}$$

Just as in the $\omega = 0$ case, the solution is not unique when $\Lambda \leqslant 0$ at the detonation front. The solution satisfying the Chapman-Jouguet condition $\Lambda = 0$ corresponds to the point of intersection of parabolas (8.2) and (8.3).

The solution for the propagation of a detonation wave in the presence of an additional spherical piston expanding from the centre is given by the integral curve of type $H_2'''C$ shown by the broken curve in Fig. 55.

The curves showing the distribution of the gas motion characteristics are given in Figs. 52, 53, and 54 in the case when the Chapman-Jouguet condition is satisfied, $\gamma = 5/3$, and $\omega = 1.5$.

If $\omega \to 2\gamma/(\gamma + 1)$, then the singular point A is displaced along parabola (8.2) and approaches the point of intersection with parabola (8.3); the points H_2 and A coincide for $\omega = 2\gamma/(\gamma + 1)$; the unique solution is obtained in which the Chapman-Jouguet condition is satisfied. If

$$\frac{2\gamma}{\gamma+1} < \omega < \frac{3(\gamma+1)}{3\gamma-1}$$

there is no solution that will satisfy the Chapman-Jouguet condition.

The solution is unique and is furnished by the integral curve starting from the singular point D and intersecting parabola (8.3) at the point H_2 for which $\Lambda > 0$ (Fig. 56).

If $\omega \to 3(\gamma + 1)/(3\gamma - 1)$, then the singular point B (see Fig. 39) moves upward, passes through the singular point A, and then exchanges roles with this point. When $\omega = 3(\gamma + 1)/(3\gamma - 1)$, the point B lies on parabola (8.3) (see Fig. 40). In this case, it is necessary to make a jump from the point O to the point B in order to obtain the solution. All the gas motions behind the wave front in the (z, V)

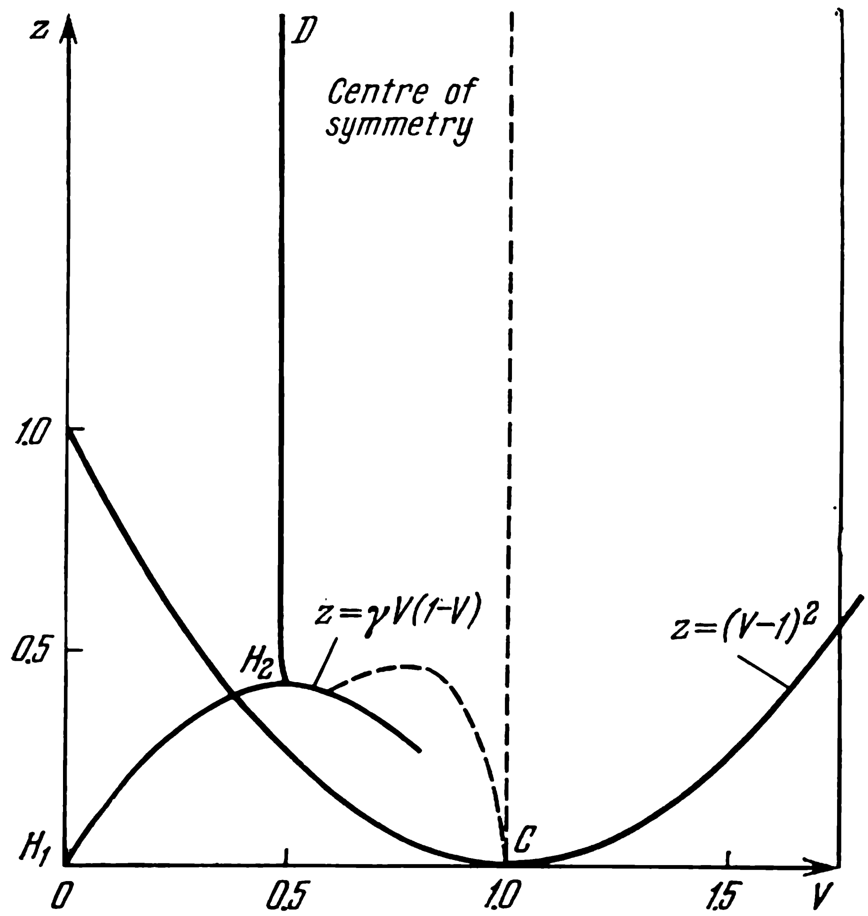

Fig. 56. The integral curve DH_2 corresponds to the solution of the detonation problem for $2\gamma/(\gamma + 1) < \omega < 3(\gamma + 1)/(3\gamma - 1)$. The Chapman-Jouguet condition is not satisfied.

plane correspond to the single point B. In this case, $z = z_2 = \text{const}$, $V = V_2 = \text{const}$ behind the wave front, and the appropriate solution is given by the simple formulas

$$\frac{v}{v_2} = \frac{r}{r_2}, \quad \frac{\rho}{\rho_2} = \frac{r_2}{r}, \quad \frac{p}{p_2} = \frac{r}{r_2} \tag{8.4}$$

The velocity behind the wave front is directly proportional to the coordinate r.

If $\omega \to 3(\gamma + 1)/(3\gamma - 1)$, then it is evident from Fig. 41 that the integral curve in the (z, V) plane passing through the point A and subsequently through the point C corresponds to the solution. An expanding sphere within which the pressure is zero corresponds to the singular point C. The value of Λ and, therefore, of the detona-

tion velocity is determined by the point of intersection of the integral curve with parabola (8.3) (Fig. 57).

The distribution of the gas characteristics is given for $\gamma = 5/3$ and $\omega = 7/3 = 2.33...$ in Figs. 52, 53, and 54.

Clearly all conclusions about the increasing detonation velocity and on the formation of a vacuum at the centre of symmetry based on self-similarity are independent of the magnitude of the liberated heat Q and only depend on the law under which the initial density falls off, which is determined by the value of the exponent ω.

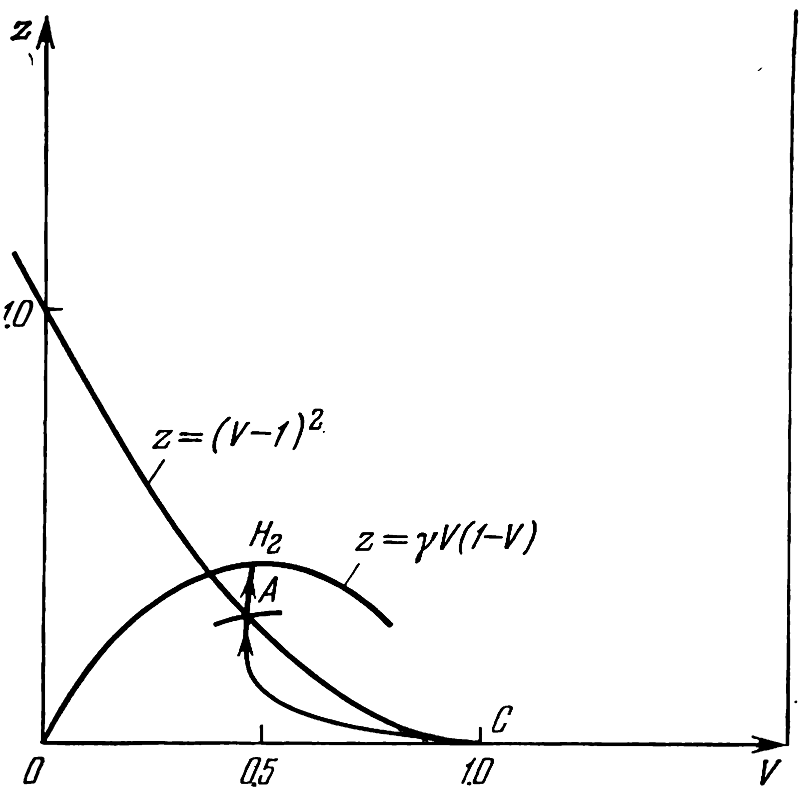

Fig. 57. The integral curve H_2AC corresponds to the solution of the detonation problem for $\omega > 3(\gamma + 1)/(3\gamma - 1)$. The point C corresponds to an expanding vacuum. The Chapman-Jouguet condition is not satisfied.

The increase in the detonation velocity as compared with the velocity given by the Chapman-Jouguet rule can be obtained also for gas detonation in tapered tubes. If the cross-sectional area varies according to a power law, then we find, using the hydraulic approximation, that the gas motion is self-similar and is determined by (5.3), (5.4), and (5.5), but only for $\nu < 1$. The magnitude of ν is determined by the law of stream tube contraction.

§ 9. Flame Propagation [27]

We consider the perturbed motion of a combustible mixture resulting from the combustion in a very thin moving layer. Analysis shows that the thickness of the layer in which a chemical reaction occurs can be neglected in a number of cases and we arrive at the problem of gas motion in which the chemical reaction and the heat liberation are performed instantaneously at a certain surface across which the variables characterizing the state and motion of the gas change discontinuously, the surface is called the flame front. In contrast to the detonation front, the flame front is a suction shock, and the velocity of propagation of the flame front, u, through the combustible mixture is a known physico-chemical constant. The flame propagation velocity is small in comparison with the speed of sound and, therefore, in comparison with the detonation velocity.

Conditions (2.12), (2.13) and (2.14) are satisfied at the flame

front, just as at the detonation front; the difference between the flame and detonation fronts is only that the flame front velocity is small and known in advance. The perturbations caused by combustion are propagated through the gas ahead of and behind the flame front. It is necessary to take the smallest root $\rho_1/\rho_2 > 1$ as the solution of (2.14). This corresponds to states which are obtained from the initial state with continuous heat liberation (without a heat absorption zone). The reaction occurs in a thin finite layer.

We consider the problem of propagation of a plane flame front through a gas at rest with the density ρ_1 and at the pressure p_1 in a cylindrical tube. Burning is ignited at the closed end of the tube. This solution of the problem is very simple and is as follows: a shock wave moves from the closed end through the unperturbed gas; behind the shock wave front the gas motion is directed forward toward the shock wave. A plane flame front is propagated through the moving gas leaving the gas behind it at rest, a consequence of the boundary conditions at the closed end. In order to solve the problem completely, it is sufficient to write and solve six equations simultaneously; three at the flame front and three at the compression shock. The six unknowns to be determined from the six equations are: the density and pressure behind the flame front and behind the compression shock, the gas velocity behind the shock wave, and the velocity of shock wave propagation.

We consider the problem of a spherical flame front under the assumption that the combustion front is ignited at $t = 0$ at a point, is then propagated by means of a spherical wave through the unperturbed gas with the constant density ρ_1 and at the constant pressure p_1. Evidently, the perturbed gas motion is self-similar and is determined by the same constants as detonation. The integral curves in the (z, V) plane for a spherical flame are given in Fig. 38 just as for the spherical detonation case.

The gas particles sufficiently far removed from the ignition centre at any instant $t > 0$ will be at rest. The stationary region corresponds to the integral line $V = 0$. It is impossible to effect the transition from rest, $V = 0$, to the motion on another integral curve in the left-hand part of the $(z > 0,\ V < 0)$ half-plane by means of a suction shock (a flame front) or through the singular point A with a weak discontinuity, since the subsequent motion cannot be continued to the centre of symmetry. In these cases, a continuous motion or motion in the presence of a shock follows an integral curve intersecting the parabola

$$z = (1 - V)^2$$

Therefore, the transition from rest, $V = 0$, to the motion on another integral curve is only possible by means of a transition through a simple compression shock originating at $z_1 < 1$. According to (2.10), compression shocks transform the $V = 0$ axis

into points of the parabola

$$z_2 = (1 - V_2)\left(1 + \frac{\gamma - 1}{2} V_2\right) \tag{9.1}$$

To continue the solution up to the centre of symmetry where $\lambda = 0$, the flame front must be so determined that a point behind it be located either on the integral line $V = 0$ leaving the singular point D ($z = \infty$, $V = 0$) or on the integral curve L entering the singular point

$$B\left(z = \frac{3(\gamma-1)^2}{(3\gamma-1)^2}, \quad V = \frac{2}{3\gamma - 1}\right)$$

(Fig. 58). (If the initial density varies according to the law $\rho_1 = A/r^{\omega}$, then the coordinates of the point B depend on ω (see § 5).) It follows from conditions (2.25) that the transition through the flame front is possible on the $V = 0$ axis only from points of the curve

$$z_3 = \frac{(1 - V_3) V_3 \left(1 + \frac{\gamma' - 1}{2} V_3\right) + (\gamma' - 1)\frac{Q}{u^2}(1 - V_3)^3}{\frac{\gamma'}{\gamma} - \frac{\gamma' - 1}{\gamma - 1}(1 - V_3)} \tag{9.2}$$

Equation (9.2) has been obtained from (2.25) after replacing the subscripts 1, 2 by 4, 3, putting $V_4 = 0$, and changing the sign in front of the quantity

$$\frac{Q}{c^2} = \frac{Q(1 - V_3)^2}{u^2}$$

since combustion occurs as the transition is made from state 3 to state 4; u is the flame propagation velocity:

$$u = c - v = \frac{r}{t}(1 - V_3) = c(1 - V_3)$$

In (9.2), γ refers to the moving combustible mixture, and γ' to the combustion products at rest.

Figure 50 shows the method of constructing the solution.

The image point is moved by the jump to a certain (z_2, V_2) point on parabola (9.1); the motion along the integral curve passing through this point in the direction of decreasing λ corresponds to a further motion toward the centre, i.e. to a motion in the region above parabola (9.1). The point of intersection of the above integral curve with curve (9.2) corresponds to the leading edge of the flame front. Behind this there is a stationary core of a gas corresponding to points of the $V = 0$ axis.

The construction described is always possible since any integral curve starting from parabola (9.1) intersects curve (9.2) for $V \leqslant 2/(\gamma + 1)$, and the transition is possible through the flame front from any point of curve (9.2) to points of the $V = 0$ axis.

However, we assume in this construction that $z_4 \geqslant 1$. Points of curve (9.2), for which $z_4 < 1$ is obtained, cannot correspond to the leading edge of the flame front since this leads to a supersonic flame front velocity relative to the particles behind the front. In this case,

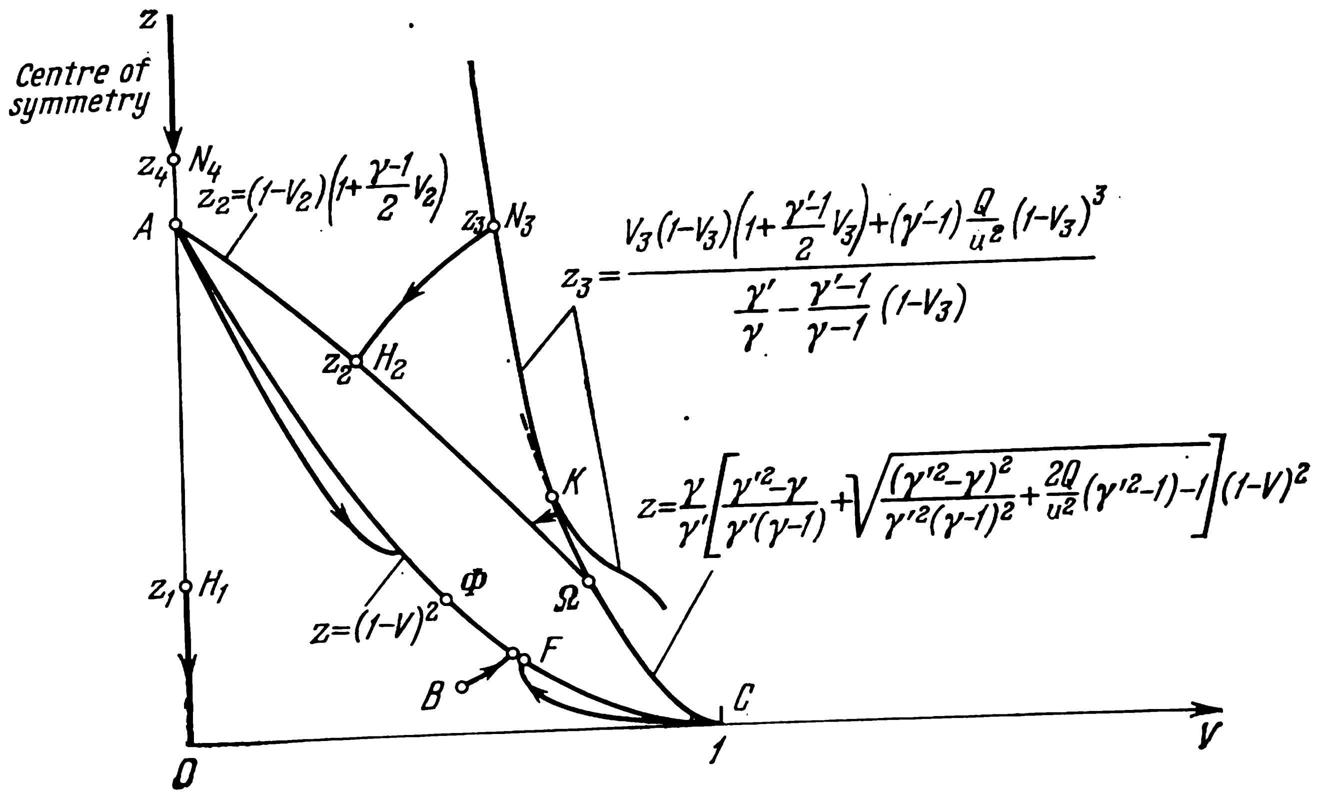

Fig. 58. The integral curves in the (z, V) plane corresponding to the spherical combustion. The points H_1 and H_2 correspond to the shock wave ahead of the flame front. The points N_3 and N_4 correspond to the flame front.

the flame front can be constructed by using a jump onto the parabola $z = (1 - V)^2$ from the point of intersection of the integral curve for the gas motion behind the shock wave with the parabola (see Fig. 58, curve $K\Omega C$)

$$z=\frac{\gamma}{\gamma'}\left[\frac{\gamma'^2-\gamma}{\gamma'(\gamma-1)}+\sqrt{\frac{(\gamma'^2-\gamma)^2}{\gamma'^2(\gamma-1)^2}+\frac{2Q}{u^2}(\gamma'^2-1)-1}\right](1-V)^2 \qquad (9.3)$$

(Parabola (9.3) transforms into the parabola $z = (1 - V)^2$ for the jump across the flame front.) The propagation velocity of such a jump through the gas behind the front exactly equals the speed of sound, and an additional suction wave is formed behind the front. This suction wave corresponds to an integral curve proceeding either from points on the parabola $z = (1 - V)^2$ to the point A, which corresponds to the boundary of the core at rest, or from the point F to the point B, in which case a core at rest is not formed, and the motion can be continued up to the centre of symmetry, or to the singular point C ($z = 0$, $V = 1$); a vacuum is formed near the centre in the latter case.

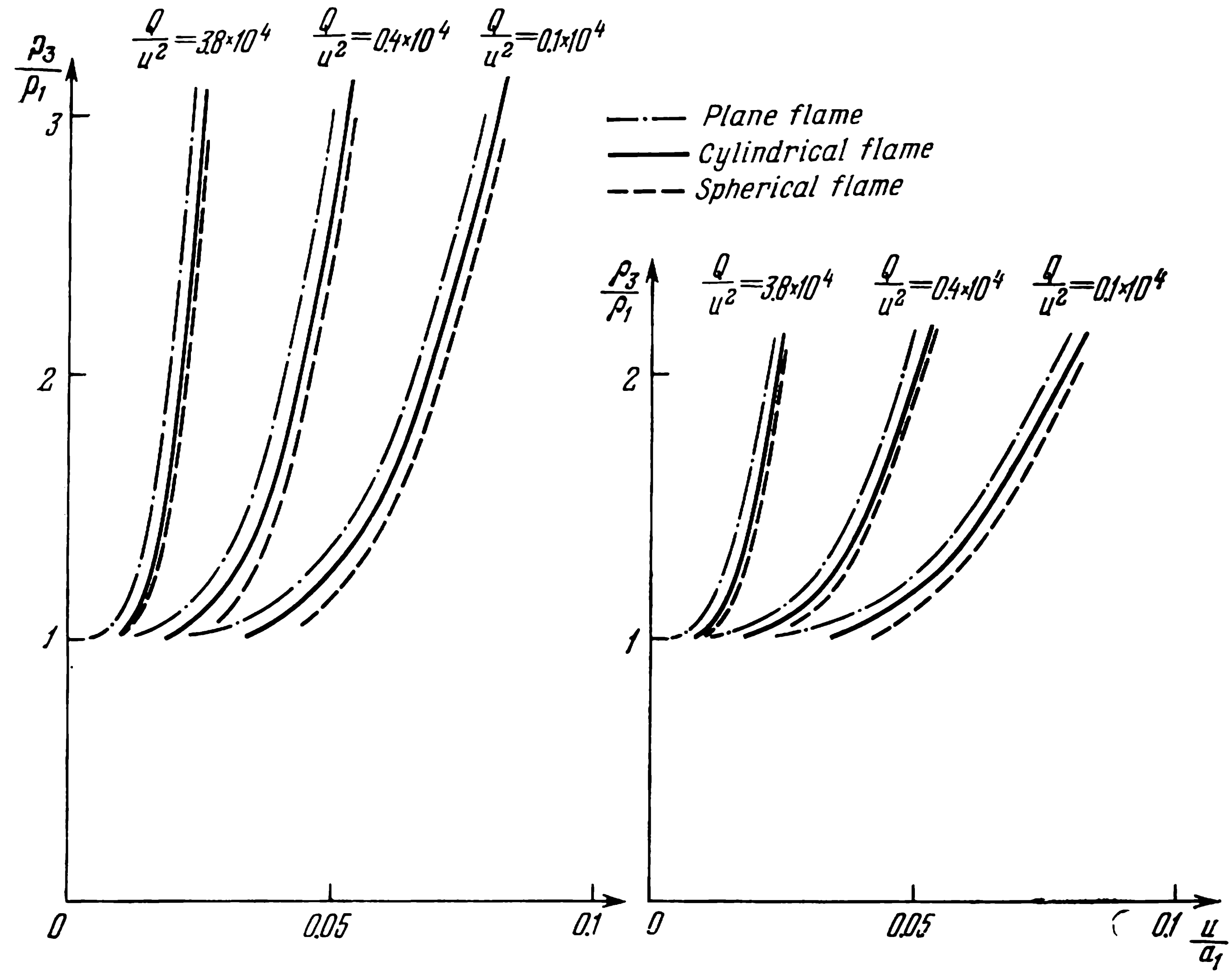

Fig. 59. Density and pressure ahead of the flame front for various amounts of heat evolved Q/u^2 (Q is the energy released per unit mass, u is the flame front velocity through the gas, p_1 is the pressure, ρ_1 is the density, and a_1 is the speed of sound in the combustible mixture).

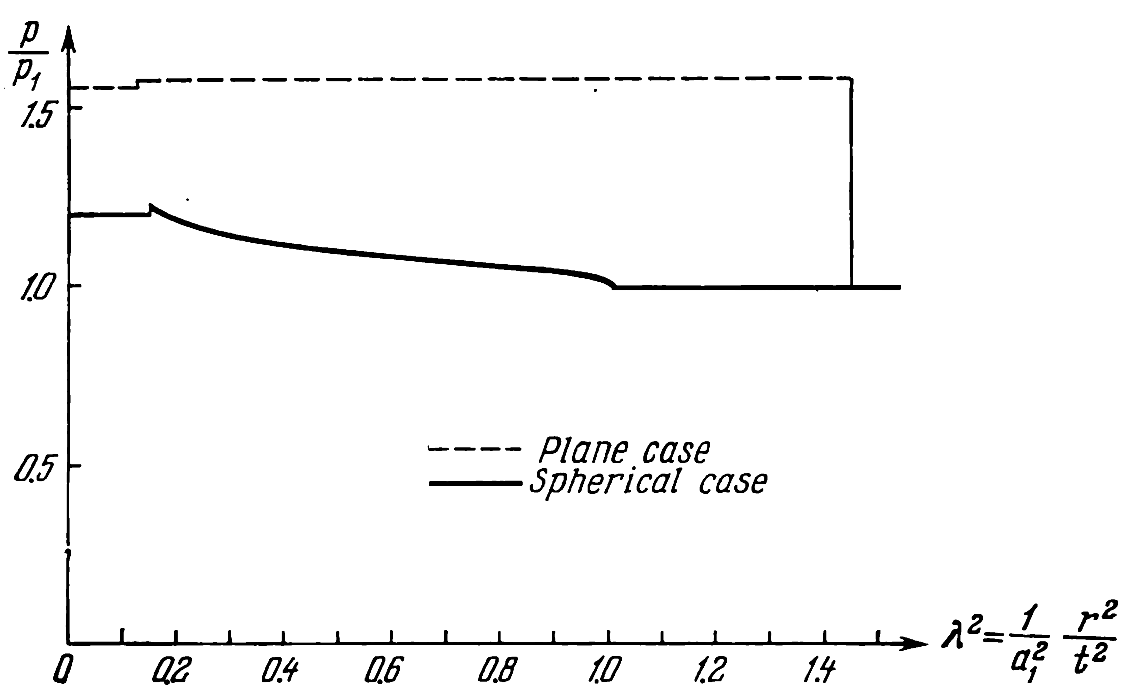

Fig. 60. Pressure distribution for propagating combustion from a plane wall (plane flame front) and from a point (spherical flame front): $Q/a_1^2 = 60$ and $u/a_1 = 0.016$.

A solution of the problem exists for all (z_2, V_2) points on parabola (9.1) located above the point Ω at which parabola (9.1) intersects parabola (9.3). The point Φ corresponds to the point Ω on the parabola $z = (1 - V)^2$. If the point Φ lies above the point F, then a stationary core is always formed near the centre of symmetry. If p_1

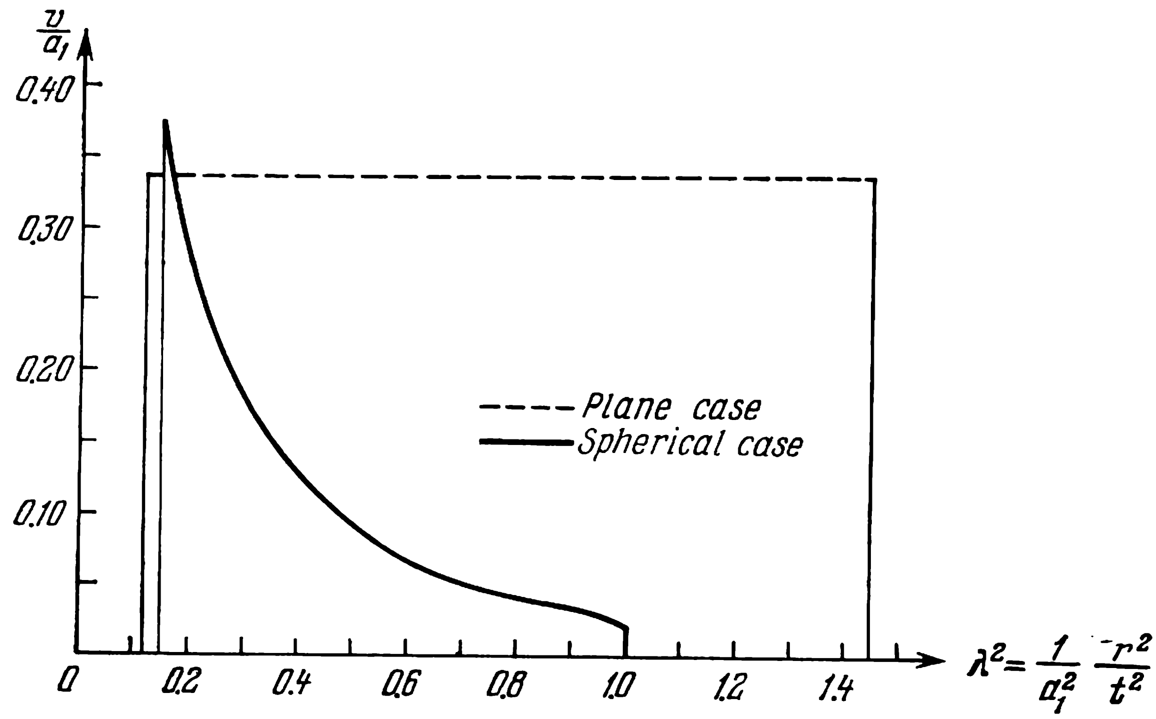

Fig. 61. Velocity distribution for propagating combustion from a plane wall (plane flame front) and from a point (spherical flame front): $Q/a_1^2 = 60$ and $u/a_1 = 0.016$.

and ρ_1 are constant, then a stationary gas core occurs near the centre of symmetry. If $\rho_1 = A/r^\omega$, a vacuum can form near the centre of symmetry for certain values of ω.

A solution of problems of propagation of cylindrical flame waves can be constructed by using similar methods. The results of numerical computations and the comparison of various cases are given in Figs. 59, 60, and 61 [28].

§ 10. Collapse of an Arbitrary Discontinuity in a Combustible Mixture

We now discuss, in outline only, the general character of the problem of the collapse of an arbitrary discontinuity (formulated in § 1). A detailed analysis will be omitted [29].

We first consider two inert gases separated by a surface of discontinuity, and assume that the pressure in the second gas to the left of the surface is larger than that in the first gas to the right of this surface (the converse case is exactly similar). Then, if the x-axis runs from left to right and if the difference $v_1 - v_2$ of the initial gas velocities is negative and large in absolute value (this case will occur, for example, if both initial gas velocities are directed toward

the surface of discontinuity), then shock waves will develop on both sides of the discontinuity. At the gas interface there is a stationary discontinuity at which the pressure and the normal velocity are continuous but the density changes discontinuously.

A graph of the pressure in this case is shown in Fig. 62*a*. As the difference in the initial velocities increases, the shock wave in the second gas changes into a suction wave (Fig. 62*c*) and then the shock wave in the first gas is replaced by a suction wave (Fig. 62*b*). When

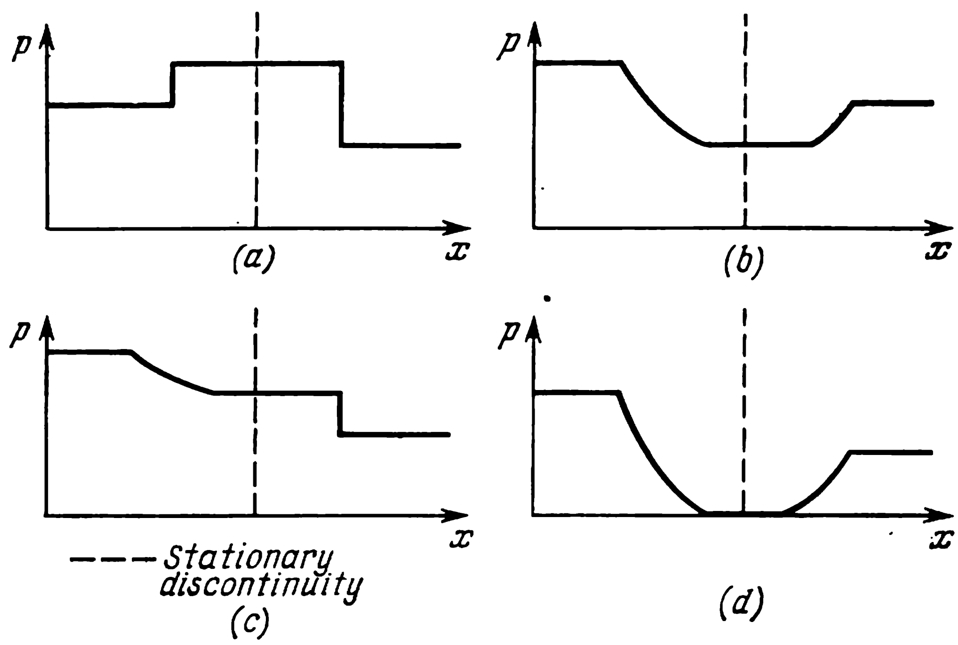

Fig. 62. Various forms of decay of an arbitrary discontinuity in an inert gas.

the initial velocity difference becomes a very large positive quantity, a vacuum forms between the suction waves on both sides (Fig. 62*d*) [30].

A more complex case arises when a combustible mixture is to the right of the surface of discontinuity so that a flame front can develop when the surface collapses arbitrarily. The general features of the motion which occurs in this case will be analogous to that considered above. (This question was worked out quantitatively in detail by Bam-Zelikovich [31].)

When the initial velocities differ by a small amount, a shock wave is propagated through an inert gas, and a shock wave also develops in a combustible mixture followed by a flame front. A stationary discontinuity can exist between the inert gas and the combustion products.

A graph of the pressure in this case is shown in Fig. 63*a* (the flame front is denoted by the wide vertical band, and the stationary discontinuity by dashes). As the difference in the initial velocities increases, the shock wave in the inert gas first changes into a suction wave (Fig. 63*c*) and then a suction wave arises ahead of the flame front in place of the shock wave (Fig. 63*b*). Here, the velocity

of the combustion products relative to the flame front increases just until sonic speed is attained.

As the difference in the initial velocities increases further, the flow ahead of the flame front does not change but still another suction wave appears directly behind the front (Fig. 63*d*). A vacuum can form between the inert gas and the combustion products if the difference in the initial velocities is very large.

If the pressure in the inert gas is less than that in the combustible mixture, a suction wave may develop in the combustible mixture ahead of the flame front and a shock wave in the inert gas.

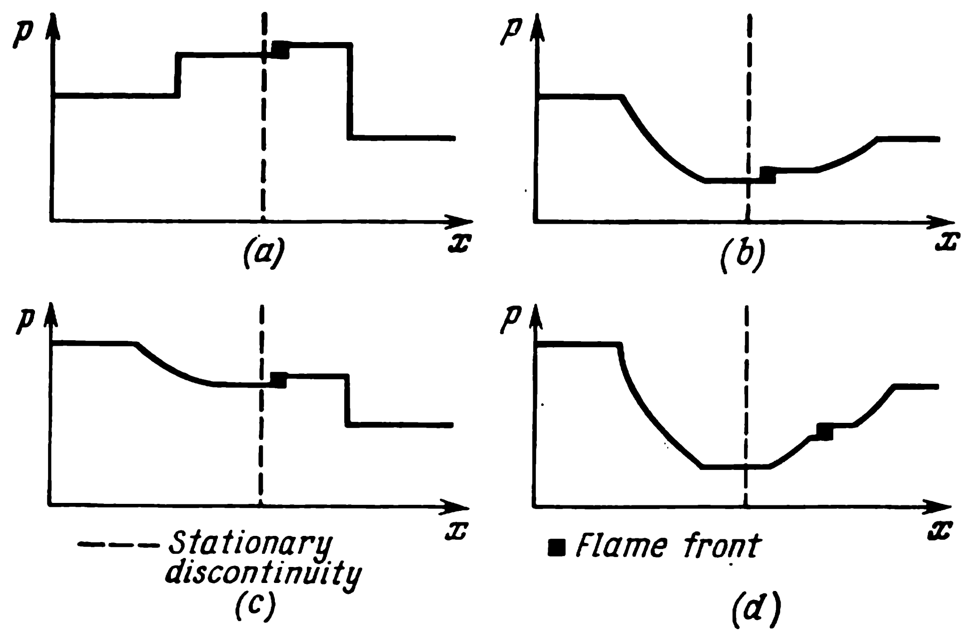

Fig. 63. Various cases of collapse of a surface of discontinuity in a combustible mixture.

Similarly, if there is a detonation wave in the combustible mixture, then by increasing the difference in the initial velocities from $-\infty$ to $+\infty$, we find that first there is a shock wave in the inert gas and a detonation wave in the combustible mixture, with an arbitrarily large velocity; then the shock wave in the inert gas is replaced by a suction wave, and the velocity of the detonation wave decreases to a certain definite value. Here the velocity of the detonation products relative to the shock front increases until the sonic speed is attained. Later, the detonation wave velocity does not change and a suction wave forms behind it.

We shall now give several types of the formation and collapse of an arbitrary discontinuity.

1. A shock wave is propagated through a gas and a second shock wave overtakes it from behind. At the instant of overtaking, a surface of discontinuity is formed across which the conditions of conservation of mass, momentum, and energy are not satisfied, i.e. an arbitrary discontinuity is formed.

Computations show that in this case after the discontinuity has collapsed the shock waves will propagate in opposite directions.

2. A shock wave approaches the interface of two media of different densities. When the shock wave crosses from one medium into the other, an arbitrary discontinuity is formed. Two types of motion are possible when this discontinuity collapses.

Shock waves will occur on both sides when the wave crosses from the less dense to the more dense medium (for example, from air to water). If the wave crosses from the more dense to the less dense medium (from water to air, say), then a shock wave propagates frontward (in air), and a suction wave rearward (in water).

3. A low-intensity shock wave overtakes a flame front. (This case is encountered in pulsating combustion in closed vessels.) After the wave has overtaken the flame, shock waves will be generated on both sides of the flame front. If a low-intensity shock wave encounters a flame front, then a suction wave will occur ahead of the flame front in the combustible mixture after an arbitrary discontinuity has collapsed, and a shock wave will develop in the combustion products.

The problem of the collapse of a given discontinuity is important in the study of the initial stages of the gas motion in shock tubes. A diagram of the motion in a shock tube is shown in Fig. 64. Two gases of high and low pressure are separated by a membrane. An arbitrary discontinuity forms after the membrane has suddenly been destroyed; consequently, a shock wave occurs in the low-pressure gas. The high-pressure gas is either at rest or is moving at the moment the membrane bursts if a shock or a detonation wave hits the membrane. The shock wave intensity in the low-pressure gas depends on the initial motion, on the pressure difference, on the temperature difference, and on the properties of the gases initially separated by the membrane.

Other conditions being equal, the shock wave intensity will increase if a gas with a reduced initial sonic speed is used as the low-pressure gas. In polyatomic gases the reduction in the sound speed can be achieved by using a gas with reduced γ. For example, at 273 K, $\gamma = 1.67$ and the speed of sound $a_1 = 975$ m/s for helium, $\gamma = 1.4$ and $a_1 = 333$ m/s for air, and $\gamma = 1.15$ and $a_1 = 121.5$ m/s for freon.

Other conditions being equal, the shock wave intensity in a low-pressure gas increases as its temperature decreases.

The shock wave intensity in a low-pressure gas is evidently very much larger than the intensity of the shock wave approaching the membrane in a high-pressure gas.

Very strong shock waves, with high temperatures behind the wave front and high-speed gas motions can be obtained in shock tubes. Particles with a high temperature which drops rapidly within a short time τ are obtained behind the shock wave front.

Shock tubes are widely used for aerodynamic investigations of very high-speed flows around bodies. They are also used in physical chemistry investigations, in particular, to obtain chemical reactions at high temperatures.

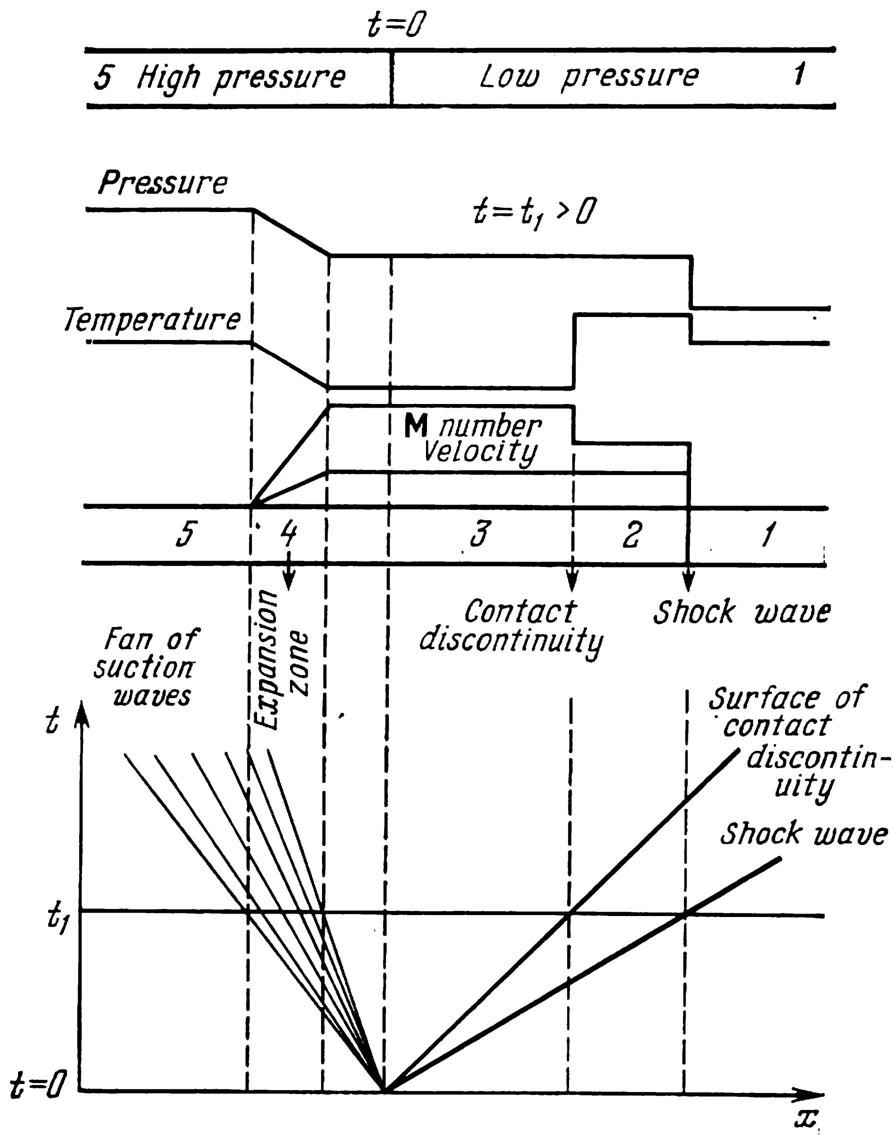

Fig. 64. Diagram of the motion in a shock tube.

The opportunity to achieve high temperatures during very short time intervals permits the kinetics of chemical reactions to be studied and intermediate products in chain reactions to be obtained.

§ 11. Problem of a Strong Explosion [32]

1. Strong Explosion in a Gas. The following arguments show that in a strong explosion the perturbed air region is separated from the unperturbed air regions by a shock wave.

As has already been mentioned, the pressure ahead of a shock wavk can be neglected in comparison with the pressure behind the shoce wave in a strong explosion. Let us first estimate with what accuracy and for which shock waves this statement is valid.

Using the property $v_1 = 0$, we rewrite shock conditions (2.5) and (2.6) as follows:

$$\left.\begin{aligned} v_2 &= \frac{2}{\gamma+1} c\left[1-\frac{a_1^2}{c^2}\right] = \frac{2c}{\gamma+1} f_1 \\ \rho_2 &= \frac{\gamma+1}{\gamma-1}\rho_1\left[1+\frac{2}{\gamma-1}\frac{a_1^2}{c^2}\right]^{-1} = \frac{\gamma+1}{\gamma-1}\rho_1 f_2 \\ p_2 &= \frac{2}{\gamma+1}\rho_1 c^2\left[1-\frac{\gamma-1}{2\gamma}\frac{a_1^2}{c^2}\right] = \frac{2}{\gamma+1}\rho_1 c^2 f_3 \end{aligned}\right\} \qquad (11.1)$$

where c is the velocity of shock wave propagation.

The more intense the shock wave the smaller the ratio a_1/c.

Figure 65 shows f_1, f_2, and f_3 as functions of a_1/c. Figure 66 shows p_2/p_1 as a function of the ratio a_1/c for $\gamma = 1.4$. We observe that the

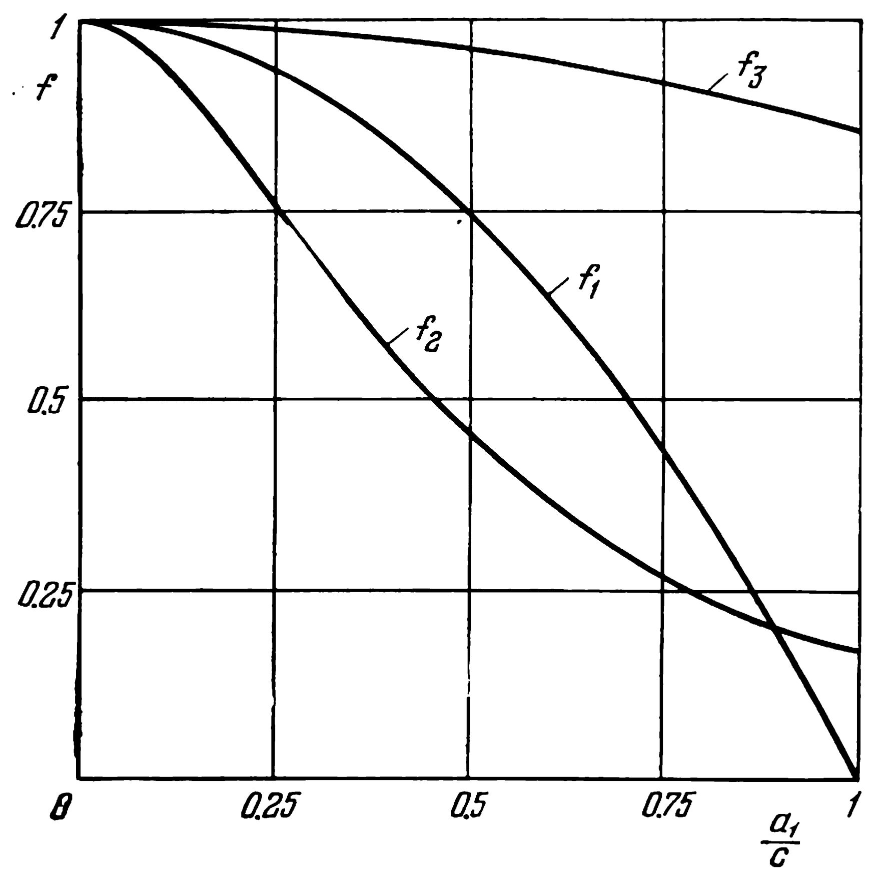

Fig. 65. Relation between the quantities f_1, f_2, and f_3 and the ratio a_1/c, where a_1 is the speed of sound in the unperturbed medium, and c is the shock wave velocity.

quantities f_1, f_2, and f_3 differ from unity by less than 5 per cent when $a_1/c < 0.1$.

If we put

$$\frac{a_1}{c} = 0 \quad \text{and} \quad f_1 = f_2 = f_3 = 1$$

into (11.1) (or if we put $p_1 = 0$, which is the same), then an error of less than 5 per cent is introduced into the values of v_2, ρ_2, and p_2.

The conditions at the shock wave then become

$$\left.\begin{aligned} v_2 &= \frac{2}{\gamma+1}c \\ \rho_2 &= \frac{\gamma+1}{\gamma-1}\rho_1 \\ p_2 &= \frac{2}{\gamma+1}\rho_1 c^2 \end{aligned}\right\} \qquad (11.2)$$

The velocity c of shock wave propagation is a characteristic parameter.

If we use the equations of motion in form (1.3) and the above formulation of the problem of a strong explosion, then we can take

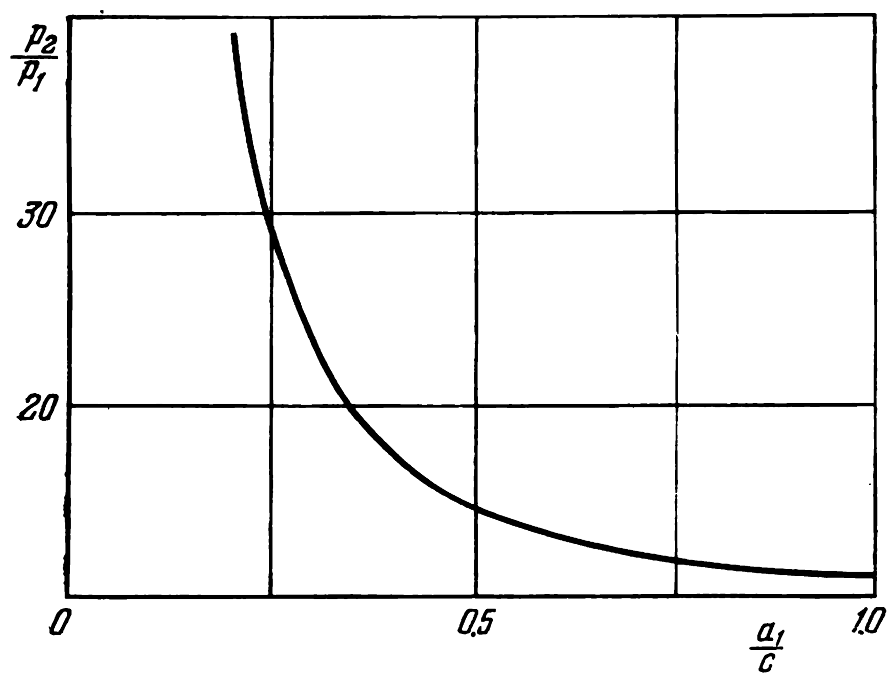

Fig. 66. Pressure drop across the shock wave as a function of the ratio a_1/c.

the following quantities as fundamental dimensional constants:

$$\rho_1 \text{ and } \frac{E}{\rho_1}$$

where E is a certain constant which we shall determine later, having the same dimensions as the energy E_0 liberated during the explosion; the dimensions of E are:

$[E] = \mathrm{ML^2T^{-2}}$ in the spherical case,
$[E] = \mathrm{MLT^{-2}}$ in the cylindrical case,
$[E] = \mathrm{MT^{-2}}$ in the plane case.

All the three cases can be combined in the single formula $[E] = \mathrm{ML^{\nu-1}T^{-2}}$.

Evidently, the constant E is directly proportional to E_0:

$$E_0 = \alpha E$$

where α is a constant.

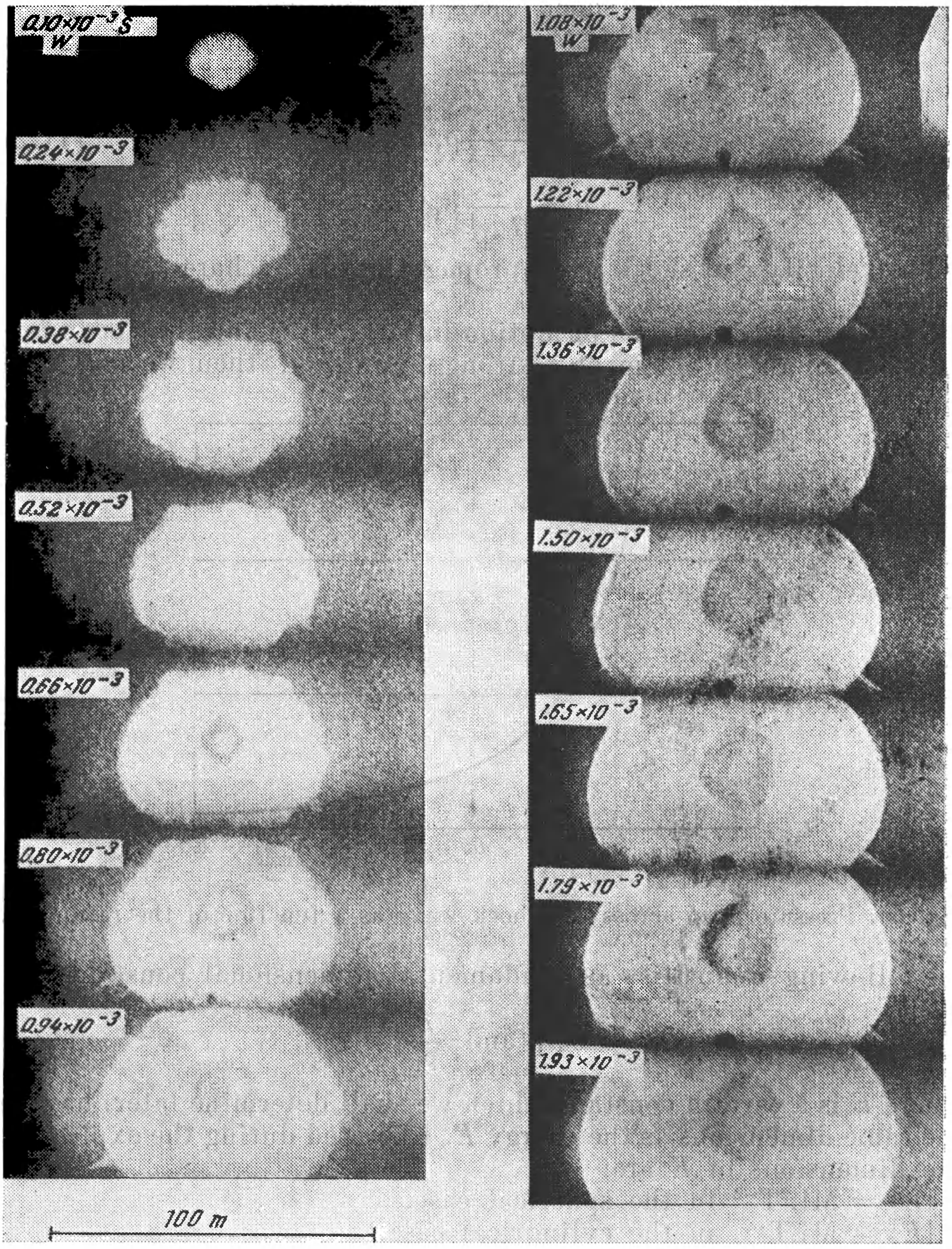

Fig. 67. Successive photographs of the fireball from $t = 0.1 \times 10^{-3}$ s to $t = 1.93 \times 10^{-3}$ s in an atomic bomb explosion in New Mexico.

In this case, the only dimensionless variable parameter λ is given by

$$\lambda = \frac{r}{\left(\frac{E}{\rho_1}\right)^{1/(2+\nu)} t^{2/(2+\nu)}}$$

The motion of a shock wave is easily determined without solving the equations of motion of a gas.

Different equations of motion can be used provided that these do not contain new essential physical constants with dimensions independent of ρ_1 and E. In particular, it is not necessary to assume that the coefficient $\gamma = c_p/c_V$ in (1.3) is constant.

The shock wave coordinate r_2 is a function of time t and, since it is impossible to form a dimensionless combination of the dimensional quantities t, ρ_1, and E,

$$r_2 = \left(\frac{E}{\rho_1}\right)^{1/(2+\nu)} t^{2/(2+\nu)} \lambda^* \tag{11.3}$$

where $\lambda^* = \text{const}$; λ^* can be set equal to any nonzero number, and the value of E can be calculated from the magnitude of the charge energy E_0. Later, to be definite and for the sake of simplicity, we shall set $\lambda^* = 1$. The constant α in the formula

$$E_0 = \alpha E$$

is then determined from the solution of the equations of motion.

Hence, in the spherical symmetry case, the motion of a shock wave is given by

$$r_2 = \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5}, \quad c = \frac{2}{5}\left(\frac{E}{\rho_1}\right)^{1/5} t^{-3/5} = \frac{2}{5}\sqrt{\frac{E}{\rho_1}}\frac{1}{\sqrt{r_2^3}} \tag{11.4}$$

and in the cylindrical symmetry case, by

$$r_2 = \left(\frac{E}{\rho_1}\right)^{1/4} \sqrt{t}, \quad c = \frac{1}{2}\left(\frac{E}{\rho_1}\right)^{1/4} \frac{1}{\sqrt{t}} = \frac{1}{2}\sqrt{\frac{E}{\rho_1}}\frac{1}{r_2} \tag{11.5}$$

while for plane waves

$$r_2 = \left(\frac{E}{\rho_1}\right)^{1/3} t^{2/3}, \quad c = \frac{2}{3}\left(\frac{E}{\rho_1}\right)^{1/3} t^{-1/3} = \frac{2}{3}\sqrt{\frac{E}{\rho_1}}\frac{1}{\sqrt{r_2}} \tag{11.6}$$

These formulas show that the law of shock wave attenuation depends on the charge shape.

The formula obtained above in the spherical symmetry case is in good agreement with published experimental data, namely, the photographs of an atomic bomb explosion in New Mexico in 1945.

The photographs of an atomic bomb explosion published in a paper by G. I. Taylor [33] are given in Figs. 67, 68, and 69.

The air temperature is very high in the perturbed air region at quite significant distances from the centre of the atomic bomb

explosion. Consequently, this region appears as a luminous spot on